Non-convex sweeping processes in contact mechanics

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Abstract

We propose a model for irreversible dynamics of the rail foundation under the effects of rail traffic, taking into account the granular structure of the ballast subject to changing void ratio and to mechanical degradation. The rail is modeled as an Euler-Bernoulli beam with distributed forcing terms representing the moving traffic load as well as the interaction with the foundation. This interaction is described by an implicit variational inequality with non-convex constraint depending in turn on the solution of the underlying PDE. The problem is reduced to a fixed point problem in a suitable Banach space, and its unique solvability is proved using the contraction principle.

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1 Introduction

Railways represent a quick and reliable means of transportation for passenger and freight. In the past years, with the increased concern about sustainability and environmental impact, many aspects of the dynamic contact problem inherent to rail transport have been investigated. In the rail track structure, the aim of the ballast bed is to provide the necessary support to the load of the moving trains while allowing for effective drainage. With such an important role in the stability of the rail system, the study of the performance of the roadbed and its deterioration mechanisms has gained a lot of attention from the engineering community, e.g. [22, 21]. In contrast to that, some aspects of mathematical modeling of railway systems are yet to be improved. Indeed, typical rail track models describe the rail as a continuous beam and the substructure as an elastic or visco-elastic foundation. See, for instance, the surveys on railway modeling [25, 15]. Although these approaches are useful for certain analysis of railway track systems, such a simplification disregard some important features of the behavior of the foundation.

The ballast is a layer of coarse grains, consisting of stones of different shapes and sizes. This particular characteristic of the ballast bed suggests that the hypoplastic concept proposed by Kolymbas to describe cohesionless granular materials, [8], presents a good framework for the investigation of the existing dynamics between the rail and the foundation. Such a theory has been successfully applied in other type of problems, even in the presence of different stiffnesses for loading and unloading cycles, which is typical for inelastic materials [2], and it opens up a new range of mathematical considerations on the rail modeling.

As an extension of the work in [7], this paper considers an Euler-Bernoulli type beam to model the dynamic behavior of a rail subjected to a moving load in contact with an inelastic foundation, see Figure 1. We assume the foundation to be a granular layer subject to structural changes resulting from the dynamic loading history. Inspired by hypoplasticity concepts and aiming accurately describe the energy exchange between the rail and the foundation, we propose a model based on the theory of non-convex sweeping processes

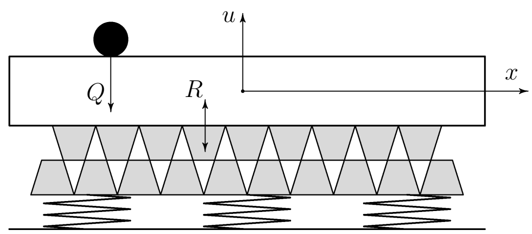


Figure 1: A scheme of the moving load along a rail on a granular foundation.

which makes clear distinction between elastic effects, evolution of the void ratio, and material degradation.

Sweeping processes are evolution problems with moving convex or non-convex constraints and they have been deeply investigated in a number of papers, e.g. [4, 5, 23, 6, 10] and the references therein. In particular, non-convex prox-regular sweeping processes, which have been widely used in crowd motion modeling [16, 17, 3], happen to offer appropriate tools to describe the motion of the ballast particles. Indeed, similarly to a highly packed situation in crowd motion modeling, the set of admissible configurations of the ballast particles is defined as a non-convex domain in $\mathbb{R}^2$, see Section 2.

The model we propose results in a coupled system of a linear elastic beam equation and a variational formulation of the sweeping processes describing the evolution of the foundation. To the best of our knowledge, this is the first time that non-convex sweeping processes coupled with a partial differential equation appear in the literature. We focus on an accurate description of the energy exchange between the load, the rail, and the foundation, benefiting from the particular characteristics of the variational setting of a sweeping process. We prove the existence and uniqueness of the solution to the problem. Notably, our proof of existence does not rely on viscous properties of the rail-foundation system. The viscosity term however could be easily included simply leading to another positive contribution to the energy balance without affecting significantly the mathematical complexity of the problem itself.

We model the rail as an elastic beam of length $2b$ and height $2h$ subjected to spatially distributed pressures R and Q. Longitudinal displacements are neglected, thus, following the 3D $\to$ 1D dimensional reduction argument in

[14], we describe the dynamics by the following PDE:

$$\rho u_{tt} - \frac{\rho h^2}{3} u_{ttxx} + \frac{Eh^2}{3} u_{xxxx} + R = Q \tag{1.1}$$

for the unknown function $u(x,t)$ representing the transversal displacement of the rail depending on $x \in (-b,b)$ and $t > 0$, where here and below, the subscripts x and t denote partial derivatives. The mixed term in Eq. (1.1), proportional to u_{ttxx}, is due to the kinetic energy of bending. We do not neglect it here, since it is of the same order of magnitude as the other terms in (1.1). The function $R = R(x,t)$ describes the contact pressure between the rail and the granular foundation, and $Q = Q(x,t)$ is the pressure induced by a given moving load (see Figure 1). Note that all terms in Eq. (1.1) have the physical dimension $[kg\, m^{-2} s^{-2}]$, that is, pressure per unit length. We focus here on the dependence of R on the history of u, and a detailed analysis is carried out in the next Section 2.

We prescribe boundary and initial conditions

$$u(-b,t) = u(b,t) = u_{xx}(-b,t) = u_{xx}(b,t) = 0 \quad \text{for} \quad t > 0, \tag{1.2}$$

$$u(x,0) = 0,\; u_t(x,0) = u^1(x) \quad \text{for} \quad x \in (-b,b). \tag{1.3}$$

The boundary conditions (1.2) characterize a simply supported beam. This is indeed not the only possible choice, and our motivation is just to make the mathematical analysis simpler. The material parameters $\rho > 0$ (mass density of the rail) and $E > 0$ (elasticity modulus) are assumed to be constant.

Roughly speaking, the contact pressure R is defined through a variational inequality which depends on a damage parameter w. In general, the degradation w depends on the loading history and leads to an implicit PDE problem. As a first step towards the solvability of such a problem, in Section 3 we show that Problem (1.1)–(1.3) has an unique solution in the case when the damage $w = w(x,t)$ is a given function. The general case, when w depends on the history of the process, is addressed in Section 4. As the main result of this paper, we prove the existence and uniqueness of solutions to Problem (1.1)–(1.3), with R defined in Section 2, under suitable assumptions on the data. The proof is based on the contraction principle combined with the local Lipschitz continuity of the input-output mapping generated by variational inequalities with convex or non-convex constraints, as proved in [9].

2 Contact with the foundation

In order to formulate the coupling conditions describing the interaction between track and foundation, this section explores elements of hypoplasticity from the perspective of sweeping processes with prox-regular constraints. To properly define the contact pressure R, we rely on the notion of *subsidence* as introduced in [20] to describe the compression of the foundation driven by gradual settling of the ballast layer interacting with the rail under the effect of dynamic loading. Following [1], we propose here to combine this idea with the concept of *void ratio* in granular materials.

At each space point x, the displacement u is additively decomposed as:

$$u = u^{rev} + u^{irr} \tag{2.1}$$

into a reversible component u^{rev} and an irreversible component u^{irr}. The evolution of the foundation is described in terms of four constitutive quantities: $s := u^{rev}$ (subsidence), $\ell := -u^{irr}$, y (history dependent irreversible part of the void ratio), and w (damage parameter), and we propose to define the set $K(w) \subset \mathbb{R}^2$ of admissible states (y, s) in terms of a function $G : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ as

$$K(w) = \left\{ \boldsymbol{\eta} = \begin{pmatrix} y \\ s \end{pmatrix} \in \mathbb{R}^+ \times \mathbb{R} : G(y, s, w) \leq 1 \right\}. \tag{2.2}$$

As long as the motion takes place in the interior of $K(w)$, the reaction of the foundation is elastic, while on the boundary of $K(w)$, irreversible processes occur. We assume the following modeling hypotheses to hold:

(i) For $s > 0$, no contact between the rail and the foundation takes place and $G(y, s, w) \leq 0$ for all $s \geq 0$;

(ii) Under constant damage w and different values $y_2 > y_1$ of the void ratio, we have $G(y_2, s, w) \geq G(y_1, s, w)$ for all $s < 0$;

(iii) Under constant void ratio y and different values $w_2 > w_1$ of the damage parameter, we have $G(y, s, w_2) \geq G(y, s, w_1)$ for all $s < 0$.

Physically, Hypothesis (ii) means that for a large void ratio, irreversible changes occur earlier than if the foundation has already undergone some compaction. Similarly, according to Hypothesis (iii), in a damaged foundation, irreversible processes start earlier than in an undamaged foundation.

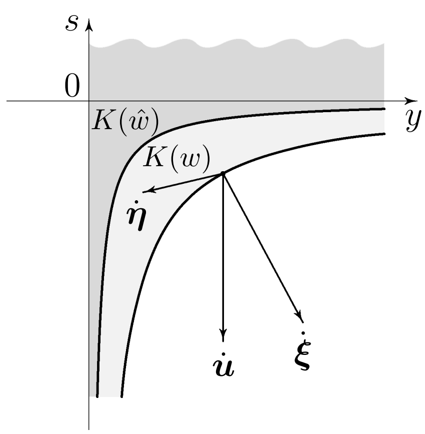
We cannot expect the admissible set $K(w)$ to be convex, and further restrictions are necessary. In order to reduce the technical complexity and keep the formulas as simple as possible, also with respect to engineering applications, we propose to make a special choice having all the required properties and consider the function G in the form

$$G(y, s, w) = -sy\,\mathrm{e}^{2w}. \tag{2.3}$$

Then the critical boundary of the admissible state domain is given by the antimonotone dependence:

$$-s_{crit} = \frac{\mathrm{e}^{-2w}}{y} \tag{2.4}$$

between the subsidence $-s_{crit}$ and the current values of y and w, see the dark gray/light gray areas in Figure 2. We shall see below in (2.10) that y can only decrease during the process. This means that during unloading, the foundation's reaction is purely elastic (see Figure 3(d)). Hence, with decreasing void ratio at constant w, the critical strain given by (2.4) at which yielding starts, increases as a result of consolidation of the material.



- $y \geq 0$ - void ratio
- s - subsidence
- $w > 0$ - damage parameter
- $K(w)$- parameter domain

Figure 2: Admissible state domain for different values $\hat{w} > w$ of the damage parameter.

In parallel to the consolidation by decreasing void ratio, we take also into account the degradation of the grains characterized by an increase of the damage parameter w, resulting from feedback produced by the accumulated contact pressure variation in (4.1) below.

The contact itself between the rail and the foundation is modeled as an input-output system, where the damage parameter $w = w(x,t)$ and the transversal displacement $u = u(x,t)$ of the rail are the inputs, and $s = s(x,t)$ and $y = y(x,t)$ are the outputs. The dependence of w on the loading history is taken into account below in (4.1), and the resulting implicit PDE problem is solved using a fixed point argument.

Let us omit for the moment the dependence on x, and consider given inputs $w : [0,T] \to [0,w^*]$ with a constant $w^* > 0$, and $u : [0,T] \to \mathbb{R}$.

The set $K(w)$ given by (2.2) with G as in (2.3) is a star-shaped $\sqrt{2}\mathrm{e}^{-w^*}$-prox-regular set. Indeed, the function G in (2.3) clearly satisfies Hypothesis 1.3 in [9] with $c = \sqrt{2}\mathrm{e}^{w}$ and $\lambda = \mathrm{e}^{2w}$, hence the fact that $K(w)$ is $\sqrt{2}\mathrm{e}^{-w^*}$-prox-regular follows from [9, Proposition 1.4]. For $t \in (0,T)$, put

$$\boldsymbol{u}(t) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad \boldsymbol{\eta}(t) = \begin{pmatrix} y(t) \\ s(t) \end{pmatrix} \in K(w(t)). \tag{2.5}$$

We define the evolution of $\boldsymbol{\eta}$ in $K(w(t))$ by the minimal action principle, which, in the non-convex context, takes the form of the variational inequality (also called a *sweeping process* according to, e. g., [18, 19, 23, 24])

$$\begin{aligned} &\left\langle \dot{\boldsymbol{\xi}}, \boldsymbol{\eta} - \boldsymbol{z} \right\rangle + \frac{\mathrm{e}^{w^*}}{2\sqrt{2}} |\dot{\boldsymbol{\xi}}||\boldsymbol{\eta} - \boldsymbol{z}|^2 \geq 0 \quad \forall \boldsymbol{z} \in K(w(t)) \quad \text{a. e.,} \\ &\boldsymbol{\xi} = \boldsymbol{u} - \boldsymbol{\eta} = \begin{pmatrix} -y \\ u - s \end{pmatrix}, \end{aligned} \tag{2.6}$$

with a given initial condition $\boldsymbol{\eta}(0) \in K(w(0))$. We choose in agreement with (1.3) initial conditions

$$w(0) = 0, \quad y(0) = y_0 \in (0,1), \quad s(0) = u(0) = 0. \tag{2.7}$$

Note that the decomposition $\dot{\boldsymbol{u}} = \dot{\boldsymbol{\xi}} + \dot{\boldsymbol{\eta}}$ represented in Figure 2 is not necessarily orthogonal on the boundary of $K(w(t))$. In general, $\dot{\boldsymbol{\xi}}(t)$ is normal to the boundary while $\dot{\boldsymbol{\eta}}(t) + \dot{w}(t)\boldsymbol{\eta}_\sharp(t)$ is tangential, where

$$\boldsymbol{\eta}_\sharp = \frac{2ys}{y^2 + s^2} \begin{pmatrix} s \\ y \end{pmatrix} \tag{2.8}$$

(see [9, Lemma 2.4]). The variational inequality (2.6) is well-posed in the sense that the input-output mapping $(w,u) \mapsto \boldsymbol{\eta}$ is well defined and continuous in both $W^{1,1}(0,T)$ and $C[0,T]$, see [9].

For several reasons, we choose the variational formulation (2.6) rather than the sweeping process setting. First, as we shall see below, it implies immediately that the system is dissipative. Second, the crucial Lipschitz continuity statement in Theorem 2.2 below is substantially based on the inequality (2.6).

Let us first derive some mathematical consequences of (2.6). For every $\delta \in (0,1)$, we have the following implication:

$$\boldsymbol{\eta} = \begin{pmatrix} y \\ s \end{pmatrix} \in K(w) \implies \boldsymbol{z} = \begin{pmatrix} (1-\delta)y \\ s \end{pmatrix} \in K(w). \tag{2.9}$$

Hence, choosing $\boldsymbol{z}$ in (2.6) in the form (2.9), dividing by δ, and letting $\delta \to 0$ we get:

$$\left\langle \dot{\boldsymbol{\xi}}, \begin{pmatrix} y \\ 0 \end{pmatrix} \right\rangle = -y\dot{y} \geq 0, \tag{2.10}$$

and we see that the void ratio is non-increasing and non-negative during the evolution. If moreover $s > 0$, then the implication (2.9) holds also for $\delta < 0$, and we conclude that $\dot{y} = 0$, meaning that the void ratio remains constant as long as $s > 0$.

Similarly, if we put in (2.6)

$$\boldsymbol{z} = \begin{pmatrix} y \\ s + \delta|s-u| \end{pmatrix}$$

for $\delta > 0$, which is indeed an admissible choice, and recalling that by (2.1) $\ell = s - u$, we get

$$\dot{\ell}(t)\delta|\ell(t)| + \frac{\mathrm{e}^{w^*}}{2\sqrt{2}}|\dot{\boldsymbol{\xi}}(t)|\delta^2|\ell(t)|^2 \geq 0.$$

Dividing by δ and letting $\delta \to 0^+$ we obtain

$$\dot{\ell}(t)|\ell(t)| \geq 0,$$

hence, the function $\ell(t)$ is nondecreasing. We shall assume that $\ell(0) = 0$ and, under this assumption, we have $\ell(t) \geq 0$ for all times.

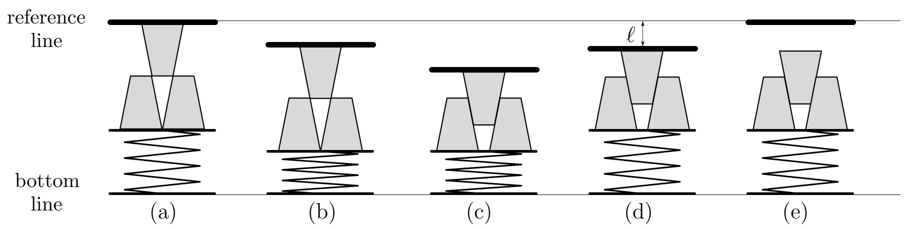


Figure 3: Reversible and irreversible behavior of the foundation.

Figure 3 illustrates a typical evolution process driven by the variational inequality (2.6). In Figure 3(a), a given point x of the rail is located on the reference line $u = 0$ with initial value $y_0 \in (0, 1)$ of the void ratio, and the foundation is in an equilibrium state. In Figure 3(b), u decreases from 0 until it reaches the critical value $u_1 = -\mathrm{e}^{-2w}/y_0$. In this interval, the foundation's reaction is elastic, and the structure of the ballast does not change. Figure 3(c) shows the case that u further decreases to some $u_2 < -\mathrm{e}^{-2w}/y_0$. Then the compactification process in the foundation starts, and the irreversible void ratio decreases to some value $0 < y_2 < y_0$. In Figure 3(d), u increases again and the irreversible void ratio remains constant, until s vanishes and u reaches the value $u_3 = -\ell$, where the elasticity of the foundation is again in equilibrium with the ballast. Then, as soon as u moves further up, as shown in Figure 3(e), the contact between the rail and the foundation is lost, and the contact pressure R vanishes, since granular materials are not extensible. Notice that this occurs when $u = -\ell$, when is $s = 0$, so contact only can take place if $s \leq 0$. The evolution diagram of the state variables in this example is represented in Figure 4 for the cases that the degradation is or is not taken into account.

Recalling Winkler's hypothesis of subgrade reaction, we know that at each point of support, the compressive force is proportional to the local subsidence (see [20]). The above analysis shows why the subsidence can be understood as active only if s is negative. This motivates the following constitutive equation:

$$R = -\mathcal{E}(y, s)s^- =: S[w, u], \tag{2.11}$$

where s^- is the negative part of s and $\mathcal{E}(y, s) > 0$ is the elasticity modulus of the foundation depending on the current values of the state variables s and y, that is, $R = \mathcal{E}(y, s)s$ for $s < 0$, $R = 0$ for $s \geq 0$.

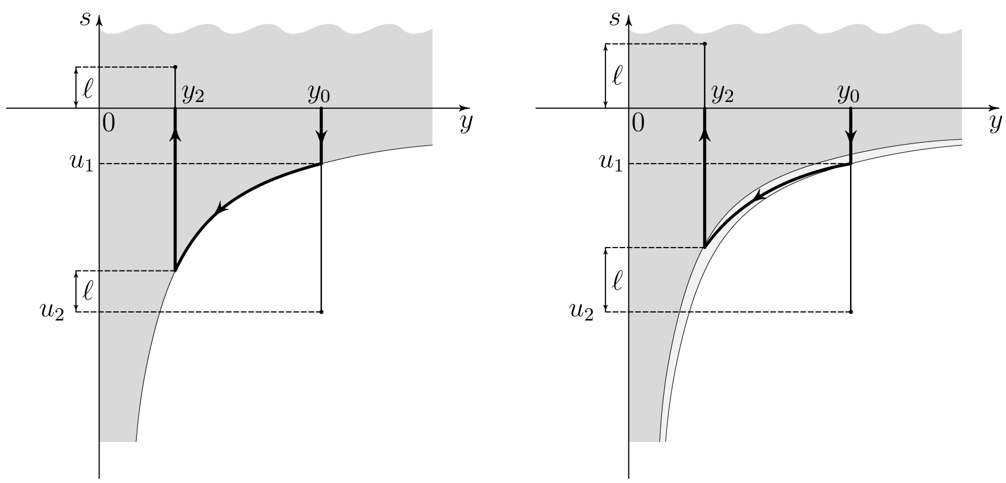


Figure 4: State evolution diagram without (left) and with (right) degradation.

This variational approach to model the rail-foundation contact admits an energetic interpretation. To see that, using the fact that $K(w)$ is star-shaped, we can choose in (2.6) $\boldsymbol{z} = (1-\delta)\boldsymbol{\eta}$ with $\delta > 0$ small, and dividing by δ we get for $\delta \to 0$ that

$$\left\langle \dot{\boldsymbol{\xi}}, \boldsymbol{\eta} \right\rangle \geq 0 \quad \text{a. e.}$$

This can be rewritten as

$$\left\langle \begin{pmatrix} -\dot{y} \\ \dot{u} - \dot{s} \end{pmatrix}, \begin{pmatrix} y \\ s \end{pmatrix} \right\rangle \geq 0.$$

Note that $\dot{\boldsymbol{\xi}} \neq 0$ only at the points of the boundary of $K(w)$, and in that case, the parameter s is necessarily negative (see also Figure 2). This means that $s = -s^-$ and the inequality above reads

$$\left\langle \begin{pmatrix} -\dot{y} \\ \dot{u} - \dot{s} \end{pmatrix}, \begin{pmatrix} y \\ -s^- \end{pmatrix} \right\rangle \geq 0,$$

that is,

$$-\dot{u}s^- \geq \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\big((s^-)^2 + y^2\big). \tag{2.12}$$

Let us choose $\mathcal{E}(y, s)$ of the form

$$\mathcal{E}(y, s) = \psi\big((s^-)^2 + y^2\big) \tag{2.13}$$

with a continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$. Recalling the definition of R in (2.11), by multiplying the inequality (2.12) by $\mathcal{E}(y,s)$ we get

$$\dot{u}(t)R(t) \geq \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(s(t), y(t)), \tag{2.14}$$

where we set

$$\mathcal{W}(y,s) = \Psi\big((s^-)^2 + y^2\big), \quad \Psi(v) = \frac{1}{2}\int_0^v \psi(z)\,\mathrm{d}z.$$

Inequality (2.14) can be interpreted as energy balance. Indeed, the left-hand side of (2.14) represents the power supplied to the system, while $\mathcal{W}(y,s)$ is the potential energy. Therefore, we can see that in an arbitrary time interval $[t_1, t_2]$ the potential energy increment is smaller than the energy supplied during this time interval, which confirms that the system is dissipative under arbitrary loading/unloading regime.

The discussion above is particularly relevant for the PDE analysis carried out in the following sections. Next, we obtain upper bounds, which will play an important role in the proof of existence of a solution in Section 4.

Proposition 2.1. *Let $u \in W^{1,\infty}(0,T)$, $w \in W^{1,1}(0,T)$, and let $U > 0$ and $w^* > 0$ be given, $|\dot{u}(t)| + |u(t)| \leq U$ a. e., $w(t) \in [0, w^*]$ for all $t \in [0,T]$. Then there exist constants $C_u, C_R > 0$ depending on U and independent of w^* such that*

$$y(t) + |s(t)| + |R(t)| \;\leq\; C_u \qquad \forall t \in [0,T] \tag{2.15}$$

$$|\dot{R}(t)| \;\leq\; C_R(1 + |\dot{w}(t)|) \qquad \text{a. e.} \tag{2.16}$$

Proof. In the variational inequality (2.6) put $\boldsymbol{z} = (1-\delta)\boldsymbol{\eta}$, divide by δ, and let $\delta \to 0$. Then

$$\left\langle \begin{pmatrix} -\dot{y} \\ \dot{u} - \dot{s} \end{pmatrix}, \begin{pmatrix} y \\ s \end{pmatrix} \right\rangle \geq 0,$$

that is,

$$\dot{u}s \geq \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\big(s^2 + y^2\big)$$

and (2.15) follows. The proof of (2.16) relies on the orthogonality of the normal and tangential vectors (cf. (2.8))

$$\left\langle \dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\eta}} + \dot{w}\boldsymbol{\eta}_\sharp \right\rangle = 0 \quad \text{a. e.}$$

This yields by virtue of (2.15) that

$$|\dot{\boldsymbol{\eta}}(t)| \leq C(1 + |\dot{w}(t)|) \quad \text{a. e.}$$

with some constant $C > 0$, which implies (2.16). ∎

Recall that the constitutive relation $R = S[w, u]$ for given functions $w : [0, T] \to [0, w^*]$, and $u : [0, T] \to \mathbb{R}$ is defined by formula (2.11), where s and y are solutions of the variational inequality (2.2)–(2.6). The local Lipschitz continuity of the input-output mapping generated by such a variational inequality has been proved in [9, Theorem 3.4]. For the reader's convenience, we state the result here.

Theorem 2.2. *Let $G : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ be a locally Lipschitz continuous function with locally Lipschitz continuous partial derivatives. Then there exist positive constants A, B, C such that for all $u_i \in W^{1,1}(0, T)$, $w_i \in W^{1,1}(0, T)$, $\boldsymbol{\eta}_i^0 \in K(w_i(0))$, $0 \leq w_i(t) \leq w^*$ for all $t \in [0, T]$, $i = 1, 2$, the solutions $\boldsymbol{\eta}_i = (y_i, s_i)$, $i = 1, 2$, to the respective variational inequalities* (2.6) *satisfy for a. e. $t \in (0, T)$ the inequality*

$$\begin{aligned}\|\dot{\boldsymbol{\eta}}_1(t) - \dot{\boldsymbol{\eta}}_2(t)\| &+ C\,\frac{\mathrm{d}}{\mathrm{d}t}\big|G(s_1(t), y_1(t), w_1(t)) - G(s_2(t), y_2(t), w_2(t))\big| \\ &\leq A\big(|\dot{u}_1(t) - \dot{u}_2(t)| + |\dot{w}_1(t) - \dot{w}_2(t)|\big) \\ &+ B\big(|\dot{u}_1(t)| + |\dot{w}_1(t)|\big)\big(|s_1(t) - s_2(t)| + |y_1(t) - y_2(t)| + |w_1(t) - w_2(t)|\big).\end{aligned}$$

The theorem above plays a key role in the prove of existence of solution for the PDE problem. In the analysis carried out in the following sections, we assume G to be the function in (2.3), which clearly satisfies the assumptions of Theorem 2.2. The extension of the results to a more general function G, although more technical, should be possible provided the selected function yields to a prox-regular set $K(w)$, and satisfies the modeling requirements as well as the hypothesis of the theorem above.

3 Existence and uniqueness of solutions to the explicit problem

We first consider the theoretical case that the function $w(x, t)$ is given, and we assume the set $K(w)$ in (2.2) with the choice G as in (2.3). The state-dependent case is solved in Section 4 by a fixed point argument.

Let us denote

$$W_0^{2,2}(-b,b) = \{\phi \in L^2(-b,b) : \ \phi_{xx} \in L^2(-b,b), \ \phi(-b) = \phi(b) = 0\}.$$

Theorem 3.1. *Let $Q \in L^2((-b,b)\times(0,T))$ and $w \in L^\infty(-b,b;W^{1,1}(0,T))$ be given functions, let $u^1 \in W^{1,2}(-b,b)$ be given, and let S be the operator defined in* (2.11)*. Then there exists a solution $u \in L^\infty(0,T;W_0^{2,2}(-b,b))$ to the PDE*

$$\begin{aligned}&\int_{-b}^{b} \left(\rho u_{tt}\phi - \frac{\rho h^2}{3}u_{tt}\phi_{xx} + \frac{Eh^2}{3}u_{xx}\phi_{xx} + S[w,u]\phi\right) \mathrm{d}x \\ &= \int_{-b}^{b} Q\phi \,\mathrm{d}x \quad \forall \phi \in W_0^{2,2}(-b,b)\end{aligned} \tag{3.1}$$

such that $u_{tx} \in L^\infty(0,T;L^2(-b,b))$, $u_{tt} \in L^2((-b,b)\times(0,T))$, satisfying the initial conditions (1.3)*, and $u(-b,t) = u(b,t) = 0$ for all $t \in (0,T)$.*

The proof will be carried out by Galerkin approximations and the argument is based on the energy balance equation.

Proof. We choose in $L^2(-b,b)$ the orthonormal system of eigenfunctions $\phi^k(x)$ associated with the problem $\phi^k_{xx} + \lambda_k\phi^k = 0$, $\phi^k(-b) = \phi^k(b) = 0$, that is,

$$\phi^k(x) = \frac{1}{\sqrt{b}}\sin\frac{k\pi}{2b}(x-b), \quad \lambda_k = \left(\frac{k\pi}{2b}\right)^2 \tag{3.2}$$

for $k \in \mathbb{N}$, and replace the PDE

$$\rho u_{tt} - \frac{\rho h^2}{3}u_{ttxx} + \frac{Eh^2}{3}u_{xxxx} + S[w,u] = Q$$

with a finite system of n ODEs for n unknown functions $u_1(t), \dots, u_n(t)$

$$\begin{aligned}&\int_{-b}^{b} \left(\rho u^{(n)}_{tt}\phi^k - \frac{\rho h^2}{3}u^{(n)}_{tt}\phi^k_{xx} + \frac{Eh^2}{3}u^{(n)}_{xx}\phi^k_{xx} + S[w,u^{(n)}]\phi^k\right) \mathrm{d}x \\ &= \int_{-b}^{b} Q\phi^k \,\mathrm{d}x,\end{aligned} \tag{3.3}$$

where $u^{(n)}(x,t) = \sum_{k=1}^{n} u_k(t)\phi^k(x)$.

Multiplying the above identity by $\dot{u}_k(t)$ and summing over k, we get the energy balance equation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-b}^{b}\left(\rho|u_t^{(n)}|^2 + \frac{\rho h^2}{3}|u_{tx}^{(n)}|^2 + \frac{Eh^2}{3}|u_{xx}^{(n)}|^2\right)\mathrm{d}x + \int_{-b}^{b} S[w,u^{(n)}]u_t^{(n)}\,\mathrm{d}x$$
$$= \int_{-b}^{b} Q u_t^{(n)}\,\mathrm{d}x.$$

From the energy balance (2.14) for the foundation it follows that the functions $u_{xx}^{(n)}$, $u_{tx}^{(n)}$ are uniformly bounded in the space $L^\infty(0,T;L^2(-b,b))$. The functions $u^{(n)}(x,t)$ admit a uniform pointwise bound

$$|u_t^{(n)}(x,t)| + |u_x^{(n)}(x,t)| \le U \tag{3.4}$$

with a constant $U > 0$ independent of n and w^*. Besides, by Sobolev embedding, the sequence $u^{(n)}$ is precompact in the space $C([-b,b]\times[0,T])$.

There exists therefore a subsequence of $\{u^{(n)}\}$ which converges uniformly to a limit $u \in C([-b,b]\times[0,T])$, and $u_{xx}^{(n)}$ converge weakly* to u_{xx} in $L^\infty(0,T;L^2(-b,b))$.

We finally test the equation (3.3) by $\ddot{u}_k(t)$ and get for $k = 1,\dots,n$ that

$$(1+k^2)|\ddot{u}_k(t)|^2 \le C\left(\tilde{Q}_k(t)|\ddot{u}_k(t)| + k^4|u_k(t)||\ddot{u}_k(t)|\right)$$

with functions $\tilde{Q}_k$ such that $\sum_{k=1}^n \int_0^T |\tilde{Q}_k(t)|^2\,\mathrm{d}t \le \tilde{C}$ for some constants $C > 0$ and $\tilde{C}$ independent of k and n. From the energy balance, it follows that $\sum_{k=1}^n k^4|u_k(t)|^2 \le C$ for all $t \in [0,T]$, hence

$$\int_0^T\int_{-b}^{b} |u_{tt}^{(n)}|^2\,\mathrm{d}x\,\mathrm{d}t = \sum_{k=0}^{n}\int_0^T |\ddot{u}_k(t)|^2\,\mathrm{d}t \le C,$$

and we may assume that $u_{tt}^{(n)}$ converge weakly to u_{tt} in $L^2((-b,b)\times(0,T))$.

We can therefore pass to the limit in Equation (3.3) and get

$$\int_{-b}^{b}\left(\rho u_{tt}\phi - \frac{\rho h^2}{3}u_{tt}\phi_{xx} + \frac{Eh^2}{3}u_{xx}\phi_{xx} + S[w,u]\phi\right)\mathrm{d}x$$
$$= \int_{-b}^{b} Q\phi\,\mathrm{d}x \quad \forall \phi \in W_0^{2,2}(-b,b),$$

for a. e. $t \in (0,T)$, which completes the existence part of the proof of Theorem 3.1.

To prove uniqueness, consider two solutions u^1, u^2 of (3.1), that is,

$$\begin{aligned}
&\int_{-b}^{b} \left(\rho u^i_{tt}\phi - \frac{\rho h^2}{3} u^i_{tt}\phi_{xx} + \frac{Eh^2}{3} u^i_{xx}\phi_{xx} + R^i\phi \right) \mathrm{d}x \\
&= \int_{-b}^{b} Q\phi \,\mathrm{d}x \quad \forall \phi \in W_0^{2,2}(-b,b),
\end{aligned} \tag{3.5}$$

with $R^i = S[w,u^i]$, $i = 1,2$ and with the same initial conditions $u^1(x,0) = u^2(x,0)$, $u^1_t(x,0) = u^2_t(x,0)$. We subtract the two equations and get (putting $\bar{u} = u^1 - u^2$, $\bar{R} = R^1 - R^2$ and omitting the constants)

$$\int_{-b}^{b} \left(\bar{u}_{tt}\phi - \bar{u}_{tt}\phi_{xx} + \bar{u}_{xx}\phi_{xx} + \bar{R}\phi \right) \mathrm{d}x = 0 \quad \forall \phi \in W_0^{2,2}(-b,b), \tag{3.6}$$

with the intention to test with $\phi = \bar{u}_t$. This is, however, not possible, as we do not control the term $\bar{u}_{xxt}$. Instead, we consider (3.6) only for $\phi = \phi^k$ with ϕ^k from (3.2), and put

$$\bar{u}_k(t) = \int_{-b}^{b} \bar{u}(x,t)\phi^k(x)\,\mathrm{d}x, \quad k = 1,\dots,n, \quad \bar{u}^{(n)}(x,t) = \sum_{k=1}^{n} \bar{u}_k(t)\phi^k(x).$$

Choosing in (3.6) $\phi = \phi^k$, testing by $\dot{\bar{u}}_k(t)$, and summing up over $k = 1,\dots,n$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-b}^{b} \left(|\bar{u}^{(n)}_t|^2 + |\bar{u}^{(n)}_{tx}|^2 + |\bar{u}^{(n)}_{xx}|^2 \right) \mathrm{d}x + \int_{-b}^{b} \bar{R}\,\bar{u}^{(n)}_t \,\mathrm{d}x = 0 \quad \forall n \in \mathbb{N}. \tag{3.7}$$

Passing to the limit as $n \to \infty$, we obtain

$$\frac{1}{2}\int_{-b}^{b} \left(|\bar{u}_t|^2 + |\bar{u}_{tx}|^2 + |\bar{u}_{xx}|^2 \right)(x,t)\,\mathrm{d}x + \int_0^t \int_{-b}^{b} \bar{R}\bar{u}_t(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \le 0,$$

which implies in particular for every $t \in (0,T)$ that

$$\int_{-b}^{b} |\bar{u}_{tx}(x,t)|^2 \,\mathrm{d}x \le C_T \int_0^t \int_{-b}^{b} \left|\bar{R}(x,\tau)\right|^2 \mathrm{d}x\,\mathrm{d}\tau \tag{3.8}$$

with a constant $C_T > 0$ depending only on the data of the problem and possibly also on T. As a consequence of (3.8), we can find possibly another constant $\tilde{C}_T$ such that for a. e. $t \in (0,T)$, we have

$$\sup_{x\in(-b,b)} \operatorname{ess} |\bar{u}_t(x,t)|^2 \le \tilde{C}_T \sup_{(x,\tau)\in(-b,b)\times(0,t)} \operatorname{ess} \left|\bar{R}(x,\tau)\right|^2 . \tag{3.9}$$

We remove the squares and rewrite (3.9) in the form:

$$\sup_{x\in(-b,b)} \operatorname{ess} |\bar{u}_t(x,t)| \le \sqrt{\tilde{C}_T} \sup_{x\in(-b,b)} \operatorname{ess} \int_0^t \left|\bar{R}_t(x,\tau)\right| \,\mathrm{d}\tau. \tag{3.10}$$

By Theorem 2.2, we have for all $x \in (-b,b)$ that

$$\begin{aligned} \int_0^t (|\bar{s}_t(x,\tau)| + |\bar{y}_t(x,\tau)|) \,\mathrm{d}\tau \le A \int_0^t |\bar{u}_t(x,\tau)| \,\mathrm{d}\tau \\ + \; B_T \int_0^t (|\bar{s}(x,\tau)| + |\bar{y}(x,\tau)|) \,\mathrm{d}\tau, \end{aligned} \tag{3.11}$$

with constants A, B_T, and by the Gronwall argument, we can rewrite this inequality as

$$\int_0^t |\bar{R}_t(x,\tau)| \,\mathrm{d}\tau \le \Gamma_T \int_0^t |\bar{u}_t(x,\tau)| \,\mathrm{d}\tau, \tag{3.12}$$

with a constant $\Gamma_T > 0$. Combining (3.10) with (3.12) and Gronwall's inequality again we obtain $\bar{u} = 0$, which completes the uniqueness proof. ■

4 State dependence

Consider now the implicit problem, where w depends on the history of the process. More specifically, assume for example, that there exists a constant $C_w > 0$ such that

$$w_t(x,t) = C_w \int_0^t |R_t(x,\tau)| \,\mathrm{d}\tau. \tag{4.1}$$

The meaning of (4.1) is that the admissible state domain $K(w)$, defined by (2.2), shrinks exponentially with the total variation of the contact pressure.

Theorem 2.2 constitutes the main argument in the state-dependent situation. Consider first the problem (3.1) with two different given functions $w^1, w^2 \in L^\infty(-b,b;W^{1,1}(0,T))$ such that

$$|w_t^i(x,t)| \le B_w \quad \text{a. e.} \tag{4.2}$$

with a constant $B_w > 0$. Let u^1, u^2 be the solutions of (3.5) with $R^i = S[w^i, u^i]$, $i = 1,2$ and with the same initial conditions $u^1(x,0) = u^2(x,0)$, $u^1_t(x,0) = u^2_t(x,0)$. We subtract the two equations and check, putting again $\bar{u} = u^1 - u^2$, $\bar{R} = R^1 - R^2$ and omitting the constants, that $\bar{u}$ and $\bar{R}$ satisfy Eq. (3.6). Repeating the argument of (3.7)–(3.9) we obtain a counterpart of (3.10) in the form

$$\sup_{x\in(-b,b)} \mathrm{ess}\, |\bar{u}_t(x,t)| \leq \sqrt{\tilde{C}_T} \sup_{x\in(-b,b)} \mathrm{ess} \int_0^t \left|\bar{R}_t(x,\tau)\right| \,\mathrm{d}\tau. \tag{4.3}$$

By Theorem 2.2, we have for all $x \in (-b,b)$ that

$$\begin{aligned} \int_0^t (|\bar{s}_t(x,\tau)| + |\bar{y}_t(x,\tau)|) \,\mathrm{d}\tau \leq A \int_0^t (|\bar{u}_t(x,\tau)| + |\bar{w}_t(x,\tau)|) \,\mathrm{d}\tau \\ + \; B_T \int_0^t (|\bar{s}(x,\tau)| + |\bar{y}(x,\tau)| + |\bar{w}(x,\tau)|) \,\mathrm{d}\tau, \end{aligned} \tag{4.4}$$

with constants A, B_T, and by the Gronwall argument, we can rewrite this inequality as

$$\int_0^t |\bar{R}_t(x,\tau)| \,\mathrm{d}\tau \leq \Gamma_T \int_0^t (|\bar{u}_t(x,\tau)| + |\bar{w}_t(x,\tau)|) \,\mathrm{d}\tau. \tag{4.5}$$

with a constant $\Gamma_T > 0$.

The solution to the implicit problem will be constructed using a fixed point argument. The solutions of (3.1) satisfy the bound (3.4) independently of w. The constant C_R introduced in (2.16) is therefore also independent of w. We define the set of admissible contact pressures

$$\begin{aligned} V := \Big\{ & R^* \in L^\infty(-b,b; W^{1,1}(0,T)) : \\ & \int_0^t |R^*_t(x,\tau)| \,\mathrm{d}\tau \leq \frac{1}{C_w} \left(\mathrm{e}^{C_R C_w t} - 1\right) \text{ a. e.} \Big\}, \end{aligned} \tag{4.6}$$

with initial condition $R^*(x,0) = -\mathcal{E}(s(x,0), y(x,0)) s^-(x,0)$, where C_w is the constant in (4.1). Assume that a function $R^* \in V$ is given, and define w by the formula

$$w(x,0) = 0, \qquad w_t(x,t) = C_w \int_0^t |R^*_t(x,\tau)| \,\mathrm{d}\tau \quad \text{a. e.} \tag{4.7}$$

By Proposition 2.1 and formula (4.7), we have

$$|R_t(x,t)| \le C_R(1+w_t(x,t)) = C_R\left(1 + C_w\int_0^t |R_t^*(x,\tau)|\,\mathrm{d}\tau\right) \le C_R \mathrm{e}^{C_R C_w t},$$

and integrating over t, we conclude that $R \in V$. Moreover, this suggests a natural choice for the constants w^* in Proposition 2.1 and B_w in (4.2), since for $R^* \in V$ and w given by (4.7), we have

$$0 \le w(x,t) \le w^* := T\left(\mathrm{e}^{C_R C_w t} - 1\right), \tag{4.8a}$$

$$0 \le w_t(x,t) \le B_w := \mathrm{e}^{C_R C_w t} - 1. \tag{4.8b}$$

Consider now the mapping $\mathcal{U}$, which, for a given $R^* \in V$ associates $R \in V$ from Theorem 3.1, with w given by (4.7).

Proposition 4.1. *There exists an equivalent norm $\|\cdot\|_*$ on the Banach space $L^\infty(-b,b;W^{1,1}(0,T))$ such that the mapping $\mathcal{U}: V \to V$ is a contraction on V with respect to this norm.*

Proof. Let $R_1^*, R_2^* \in V$ be given, let u_1, u_2 be the corresponding solutions to (3.1) and with w_1, w_2 associated with R_1^*, R_2^* according to (4.7) and with the same initial conditions. Let $\bar{u} = u_1 - u_2$, $\bar{u} = u_1 - u_2$, $\bar{s} = s_1 - s_2$, $\bar{y} = y_1 - y_2$, $\bar{w} = w_1 - w_2$, $\bar{R} = R_1 - R_2$, $\bar{R}^* = R_1^* - R_2^*$.

By (4.3) and (4.5), we have for a. e. $x \in (-b,b)$ that

$$\begin{aligned} \int_0^t |\bar{R}_t(x,\tau)|\,\mathrm{d}\tau &\le C\int_0^t \left(|\bar{u}_t(x,\tau)| + |\bar{w}_t(x,\tau)|\right)\,\mathrm{d}\tau \\ &\le C_T^* \int_0^t \operatorname*{sup\,ess}_{x\in(-b,b)} \int_0^\tau |\bar{R}_t^*(x,\rho)|\,\mathrm{d}\rho\,\mathrm{d}\tau \end{aligned} \tag{4.9}$$

with constants $C > 0, C_T^* > 0$ depending only on the data. Put

$$v(t) = \operatorname*{sup\,ess}_{x\in(-b,b)} \int_0^t |\bar{R}_t(x,\tau)|\,\mathrm{d}\tau, \quad v^*(t) = \operatorname*{sup\,ess}_{x\in(-b,b)} \int_0^t |\bar{R}_t^*(x,\tau)|\,\mathrm{d}\tau, \quad M := 2C_T^*.$$

Integrating by parts and using (4.9), we get

$$\begin{aligned}
&\int_0^T \mathrm{e}^{-Mt} \int_0^t v(\tau)\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \frac{1}{M}\left[-\mathrm{e}^{-Mt}\int_0^t v(\tau)\,\mathrm{d}\tau\right]_{t=0}^T + \frac{1}{M}\int_0^T \mathrm{e}^{-Mt} v(t)\,\mathrm{d}t \\
&\leq \frac{1}{2}\int_0^T \mathrm{e}^{-Mt}\int_0^t v^*(\tau)\,\mathrm{d}\tau\,\mathrm{d}t.
\end{aligned}$$

We see that the mapping $\mathcal{U} : V \to V$, which with R^* associates R, is a contraction with respect to the norm $\|\cdot\|_*$ in $L^\infty(-b,b;W^{1,1}(0,T))$, defined as

$$\|R\|_* := \left(\operatorname*{sup\,ess}_{x\in(-b,b)} |R(x,0)| + \int_0^T \mathrm{e}^{-Mt}\int_0^t \operatorname*{sup\,ess}_{x\in(-b,b)} \int_0^\tau |R_t(x,\rho)|\,\mathrm{d}\rho\,\mathrm{d}\tau\,\mathrm{d}t\right),$$

more specifically,

$$\|R_1 - R_2\|_* \leq \frac{1}{2}\|R_1^* - R_2^*\|_*, \tag{4.10}$$

which completes the proof. ■

Theorem 4.2. *Let w be of the form (4.1). Then, there exists a unique solution u to the PDE*

$$\begin{aligned}
&\int_{-b}^b \left(\rho u_{tt}\phi - \frac{\rho h^2}{3}u_{tt}\phi_{xx} + \frac{Eh^2}{3}u_{xx}\phi_{xx} + S[w,u]\phi\right)\,\mathrm{d}x \\
&= \int_{-b}^b Q\phi\,\mathrm{d}x \quad \forall \phi \in W_0^{2,2}(-b,b)
\end{aligned} \tag{4.11}$$

such that $u_{tx}, u_{xx} \in L^\infty(0,T;L^2(-b,b))$, $u_{tt} \in L^2((-b,b)\times(0,T))$, satisfying the initial conditions, and $u(-b,t) = u(b,t) = 0$ for all $t \in (0,T)$.

ee4

Proof. Put $R^{(0)}(x,t) = R(x,0) = -\mathcal{E}(y(x,0),s(x,0))s^-(x,0)$. For $j \in \mathbb{N}$, we define recursively according to (4.7) and (2.11) the sequence

$$w^{(j)}(x,t) = C_w\int_0^t (t-\tau)|R_t^{(j-1)}(x,\tau)|\,\mathrm{d}\tau, \quad R^{(j)} = S[w^{(j)},u^{(j)}],$$

where $u^{(j)}$ is the solution of the equation

$$\int_{-b}^{b} \left(\rho u_{tt}^{(j)} \phi - \frac{\rho h^2}{3} u_{tt}^{(j)} \phi_{xx} + \frac{E h^2}{3} u_{xx}^{(j)} \phi_{xx} + S[w^{(j)}, u^{(j)}] \phi \right) \mathrm{d}x$$
$$= \int_{-b}^{b} Q \phi \, \mathrm{d}x \quad \forall \phi \in W_0^{2,2}(-b, b).$$

From (4.10) it follows that $\{R^{(j)} : j \in \mathbb{N}\}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_*$, hence, it admits a limit $R \in V$. From (3.8), we conclude that $\{u^{(j)}\}$ converge to the unique solution u to (4.11), which completes the proof of Theorem 4.2. ■

Conclusions and further perspectives

We propose a new rail track model in which the dynamic interaction between the rail and the granular foundation is described via a variational inequality with non-convex constraint defined as the sublevel set of a function G given by (2.3). To our knowledge, this is the first publication combining the theories of non-convex sweeping processes and partial differential equations with applications in contact mechanics. There is a lot of space for possible generalizations including for example elastoplastic bodies, or less restrictive conditions on the function G, or on the damage accumulation law (4.1). We plan to focus on these problems in the future.

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