THE INTERFACE CRACK WITH COULOMB FRICTION BETWEEN TWO BONDED DISSIMILAR ELASTIC MEDIA*

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Dedicated to Professor K. R. Rajagopal on the occasion of his 60th birthday

Abstract. We study a model of interfacial crack between two bonded dissimilar linearized elastic media. The Coulomb friction law and non-penetration condition are assumed to hold on the whole crack surface. We define a weak formulation of the problem in the primal form and get the equivalent primal-dual formulation. Then we state the existence theorem of the solution. Further, by means of Goursat-Kolosov-Muskhelishvili stress functions we derive convergent expansions of the solution near the crack tip.

Keywords: linearized elasticity, singularities at the crack tip, interfacial crack, non-penetration condition, Coulomb friction

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1. Introduction

We consider the problem of the non-ideal bond between two dissimilar linearized elastic media allowing for a crack between them. By this, we assume that the friction is possible between the crack faces being in contact. We describe the friction with the Coulomb law.

The principal difficulty of the model concerns the friction condition near the crack tip where the main singularity occurs. For comparison, for the contact of two bodies the friction condition can be separated from the end point of the contact boundary

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thus avoiding the geometric singularity. The classical framework of Coulomb friction model can be found in [23], [25] and other works. For modelling of frictional cracks we refer to [2], [4], [5], [6], [7], [30]. The seminal work [28] provided a method that made it possible to start the studies of such problems just 30 years ago.

We investigate the problem in the weak formulation written in two equivalent forms in Section 3. First, the pure primal formulation provides us with the common quasi-variational inequality. Second, the primal-dual formulation accounts for the displacement and the stress at the crack as independent variables. The mathematical difficulty lies in the fact that the problem cannot be expressed as the minimization problem with respect to the elastic potential energy. Therefore, one of the principal questions of our investigation is the existence of the solution.

For an overview of available techniques adopted in the field of frictional problems we refer to the books [8], [31] and the references therein. The common assumptions which guarantee the existence are that the friction coefficient is sufficiently small, and it has a compact support, in our case, in the crack. While the latter assumption was used in [24], in the present paper we avoid this restrictive assumption using the topological sensitivity technique developed recently in [20], [21], [22] for the constrained crack problems. The principal estimate is associated with the Saint-Venant principle. For investigation of multiplicity of the solution we refer to [12], [29] and to [11] for its bifurcation.

Further, in Section 4 we get the asymptotic expansion of the solution in the vicinity of the crack tip under each one of three conditions: open crack, stick state, slip state. Singularity of the special solution for such kinds of the problem has been well studied in engineering (e.g. [5]), however, to our knowledge, it has remained an open problem whether all weak solutions have such asymptotic expansions. Then by means of Goursat-Kolosov-Muskhelishvili stress functions we verify that exactly by the convergence proof. At the same time, it provides us with the a priori regularity of the solution.

2. Formulation of the problem

Let Ω be a bounded domain of $\mathbb{R}^2$ with Lipschitz boundary and divided into two parts $\Omega^{(1)}$ and $\Omega^{(2)}$ by the x_1-axis, that is, $\Omega^{(1)} = \Omega \cap \{x_2 > 0\}$ and $\Omega^{(2)} = \Omega \cap \{x_2 < 0\}$. Let both $\Omega^{(1)}$ and $\Omega^{(2)}$ be Lipschitz domains. Each $\Omega^{(k)}$ $(k = 1, 2)$ represents a dissimilar isotropic homogeneous linearized elasticity. We denote the interface of $\Omega^{(k)}$ by Γ'. Let Γ be a crack lying on the interface Γ' and having two crack tips located at the origin $\boldsymbol{O} \notin \partial\Omega$ of the coordinate system $\boldsymbol{x} = (x_1, x_2)$ and at a point $\boldsymbol{P}(-l, 0) \in \partial\Omega$, $l > 0$, see Fig. 1 for an illustration of the geometry.

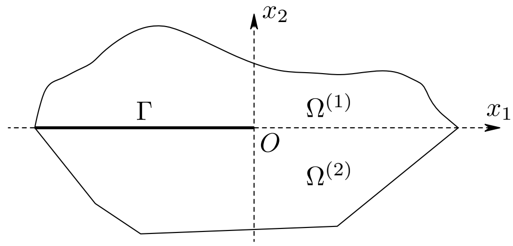

Figure 1. The domain Ω.

By $\boldsymbol{u}^{(k)} = (u_i^{(k)})_{i=1,2}$ and $\boldsymbol{\sigma}^{(k)} = (\sigma_{ij}^{(k)})_{i,j=1,2}$ we denote the displacement vector and the stress tensor, respectively. The superscripts $k = 1$ and $k = 2$ refer to the materials in $\Omega^{(1)}$ and $\Omega^{(2)}$, respectively. Throughout the paper, we denote a generic positive constant by c.

We introduce the jump of $\boldsymbol{u}$ at Γ' by the formula

$$[\boldsymbol{u}] := \boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)} \quad \text{on } \Gamma'.$$

In each $\Omega^{(k)}$ we suppose the stationary equilibrium conditions without any body forces hold, which are described as

$$\frac{\partial}{\partial x_j}\sigma_{ij}^{(k)} = 0, \quad i = 1, 2. \tag{2.1}$$

Then, the linearized elasticity equations for $\boldsymbol{u}^{(k)}$ are given by

$$A^{(k)}\boldsymbol{u}^{(k)} := \mu^{(k)}\Delta\boldsymbol{u}^{(k)} + (\tilde{\lambda}^{(k)} + \mu^{(k)})\nabla(\nabla\cdot\boldsymbol{u}^{(k)}) = \boldsymbol{0} \quad \text{in } \Omega^{(k)}.$$

Here and in what follows we use the summation convention,

$$\tilde{\lambda}^{(k)} = \begin{cases} \lambda^{(k)} & \text{(plane strain)}, \\ \dfrac{2\lambda^{(k)}\mu^{(k)}}{\lambda^{(k)} + 2\mu^{(k)}} & \text{(plane stress)}, \end{cases}$$

$\lambda^{(k)}$ and $\mu^{(k)}$ are the *Lamé* constants of the two elastic media, respectively. Since both the shear modulus and the bulk modulus are required to be positive, we suppose $\mu^{(k)} > 0$ and $\lambda^{(k)} + \mu^{(k)} > 0$, in which case it is easy to see that the operator $A^{(k)}$ is elliptic. And we define $\tilde{\kappa}^{(k)} = (\tilde{\lambda}^{(k)} + 3\mu^{(k)})/(\tilde{\lambda}^{(k)} + \mu^{(k)})$. Moreover, we introduce the boundary stress operator T and the stress vector $T\boldsymbol{u}^{(k)}$ expressed by $T\boldsymbol{u}^{(k)} := \boldsymbol{\sigma}^{(k)}\boldsymbol{n}$, where $\boldsymbol{n} = (n_1, n_2)$ is the unit outward normal vector field on $\partial\Omega$ and

$$\boldsymbol{\sigma}^{(k)} = \tilde{\lambda}^{(k)}(\nabla\cdot\boldsymbol{u}^{(k)})\boldsymbol{I} + \mu^{(k)}\{\nabla\boldsymbol{u}^{(k)} + (\nabla\boldsymbol{u}^{(k)})^{\mathrm{T}}\}, \tag{2.2}$$

where $\boldsymbol{I}$ is the second order identity tensor.

Now we consider the following boundary value problem $(*)$: for given $\boldsymbol{g} \in L^2(\partial\Omega)$ such that $\boldsymbol{g} = \boldsymbol{0}$ near $\boldsymbol{P}$, and a small constant friction coefficient $f > 0$ (see (3.11)), find $\boldsymbol{u}^{(1)} \in H^1(\Omega^{(1)})$ and $\boldsymbol{u}^{(2)} \in H^1(\Omega^{(2)})$ satisfying

$$(*) \quad \begin{cases} A^{(1)}\boldsymbol{u}^{(1)} = \boldsymbol{0} & \text{in } \Omega^{(1)}, \\ A^{(2)}\boldsymbol{u}^{(2)} = \boldsymbol{0} & \text{in } \Omega^{(2)}, \\ T\boldsymbol{u}^{(1)} = \boldsymbol{g} & \text{on } \partial\Omega^{(1)} \cap \partial\Omega, \\ T\boldsymbol{u}^{(2)} = \boldsymbol{g} & \text{on } \partial\Omega^{(2)} \cap \partial\Omega, \\ [u_1] = [u_2] = [\sigma_{12}] = [\sigma_{22}] = 0 & \text{on } \Gamma' \setminus \overline{\Gamma}, \\ [\sigma_{22}] = 0, \quad \sigma_{22}^{(k)} \leqslant 0, \quad [u_2] \geqslant 0, \quad \sigma_{22}^{(k)}[u_2] = 0 & \text{on } \Gamma, \\ [\sigma_{12}] = 0, \quad |\sigma_{12}^{(k)}| \leqslant -f\sigma_{22}^{(k)}, \quad \sigma_{12}^{(k)}[u_1] + f\sigma_{22}^{(k)}|[u_1]| = 0 & \text{on } \Gamma. \end{cases}$$

Note that we model the Neumann conditions on $\partial\Omega$. The Dirichlet and mixed boundary conditions can be treated within our approach in a similar manner.

In the problem $(*)$ the boundary conditions on Γ include the following three cases:

(1) $[u_2] > 0$ on Γ (open crack).
In this case they can be reduced to

$$\sigma_{12}^{(k)} = \sigma_{22}^{(k)} = 0 \quad \text{on } \Gamma. \tag{2.3}$$

(2) $[u_2] = 0$ on Γ.

(a) $[u_1] = 0$ on Γ (stick state).
In this case they can be reduced to

$$[\sigma_{22}] = [\sigma_{12}] = 0 \quad \text{on } \Gamma, \tag{2.4}$$

$$\sigma_{22}^{(k)} \leqslant 0 \quad \text{on } \Gamma, \tag{2.5}$$

$$|\sigma_{12}^{(k)}| \leqslant -f\sigma_{22}^{(k)} \quad \text{on } \Gamma. \tag{2.6}$$

(b) $[u_1] \neq 0$ on Γ (slip state).
In this case they can be reduced to

$$[\sigma_{22}] = [\sigma_{12}] = 0 \quad \text{on } \Gamma, \tag{2.7}$$

$$\sigma_{22}^{(k)} \leqslant 0 \quad \text{on } \Gamma, \tag{2.8}$$

$$\sigma_{12}^{(k)} \pm f\sigma_{22}^{(k)} = 0 \quad \text{on } \Gamma, \tag{2.9}$$

where the upper sign "+" is taken for $[u_1] > 0$ on Γ and the lower sign "−" is taken for $[u_1] < 0$ on Γ.

We justify conditions (2.3)–(2.9) using projections. To this aim we introduce the closed convex set

$$M = \{\boldsymbol{p} = (p_1, p_2) \in L^\infty(\Gamma)\colon\ |p_1| \leqslant -f p_2\}.$$

Recalling that $[\sigma_{22}] = [\sigma_{12}] = 0$ on Γ', for $\sigma_{12} = \sigma_{12}^{(k)}$ and $\sigma_{22} = \sigma_{22}^{(k)}$ the boundary conditions on Γ in $(*)$ yield the dual form

$$\sigma_{22} \leqslant 0, \quad (p_2 - \sigma_{22})[u_2] \leqslant 0 \quad \forall\, p_2 \leqslant 0, \tag{2.10}$$

$$(\sigma_{12}, \sigma_{22}) \in M, \quad (p_1 - \sigma_{12})[u_1] + f(p_2 - \sigma_{22})|[u_1]| \leqslant 0 \quad \forall\, \boldsymbol{p} \in M. \tag{2.11}$$

Multiplying (2.10) and (2.11) by arbitrary constants $a > 0$ and $b > 0$, we obtain

$$(\sigma_{22} - p_2)\big((\sigma_{22} + a[u_2]) - \sigma_{22}\big) \geqslant 0 \quad \forall\, p_2 \leqslant 0,$$
$$((\sigma_{12}, \sigma_{22}) - \boldsymbol{p}) \cdot \big((\sigma_{12} + b[u_1], \sigma_{22} + bf|[u_1]|) - (\sigma_{12}, \sigma_{22})\big) \geqslant 0 \quad \forall\, \boldsymbol{p} \in M,$$

which implies the projections onto $\mathbb{R}^-$ and M, respectively, that is,

$$\sigma_{22} = -\{\sigma_{22} + a[u_2]\}^-, \quad (\sigma_{12}, \sigma_{22}) = \pi_M(\sigma_{12} + b[u_1], \sigma_{22} + bf|[u_1]|), \tag{2.12}$$

with the notation $-\{\xi\}^- = \min(0, \xi)$. Given $\boldsymbol{u}$ the system (2.12) provides three equations for two unknowns σ_{12} and σ_{22}. They are compatible by setting the specific projection operator $\pi_M\colon\ L^\infty(\Gamma) \mapsto M$ by

$$\pi_M p_1 = p_1 - \{f\{p_2\}^- - p_1\}^- + \{f\{p_2\}^- + p_1\}^-, \quad \pi_M p_2 = -\{p_2\}^-.$$

As the result, from (2.12) we arrive at the following two projection equations:

$$\sigma_{22} = -\{\sigma_{22} + a[u_2]\}^-, \tag{2.13}$$

$$\begin{aligned} 0 = b[u_1] &- \big\{f\{\sigma_{22} + bf|[u_1]|\}^- - \sigma_{12} - b[u_1]\big\}^- \\ &+ \big\{f\{\sigma_{22} + bf|[u_1]|\}^- + \sigma_{12} + b[u_1]\big\}^-. \end{aligned} \tag{2.14}$$

System (2.13)–(2.14) realizes (2.3)–(2.9). Indeed, we check these conditions:

(1) On the inactive set of points in Γ such that $\sigma_{22} + a[u_2] > 0$, from (2.13) we obtain $\sigma_{22} = 0$ and $[u_2] > 0$. Thus the crack is open. Equation (2.14) implies $0 = b[u_1] - \{-\sigma_{12} - b[u_1]\}^- + \{\sigma_{12} + b[u_1]\}^- = -\sigma_{12}$ and, therefore, conditions (2.3).

(2) On the complementary active set, where $\sigma_{22} + a[u_2] \leqslant 0$, from (2.13) we derive $[u_2] = 0$ and $\sigma_{22} \leqslant 0$, thus the crack is closed.

   (a) On the subset of the active set, where $|\sigma_{12} + b[u_1]| \leqslant f\{\sigma_{22} + bf|[u_1]|\}^-$, equation (2.14) yields $0 = b[u_1]$ and the stick conditions (2.5)–(2.6).

(b) On its complementary subset, from (2.14) we conclude either $\sigma_{12} = f\{\sigma_{22} + bf|[u_1]|\}^-$ and $[u_1] > 0$ for $\sigma_{12} + b[u_1] > f\{\sigma_{22} + bf|[u_1]|\}^-$, or $\sigma_{12} = -f\{\sigma_{22} + bf|[u_1]|\}^-$ and $[u_1] < 0$ for $\sigma_{12} + b[u_1] < -f\{\sigma_{22} + bf|[u_1]|\}^-$. It yields exactly the slip conditions (2.8)–(2.9).

This representation is useful for the approximation of problem $(*)$, see the related topic in [13], [14].

3. The weak solution and the regularity

In order to provide the boundary stress with an exact meaning we employ the Green formulae written in the Lipschitz domains $\Omega^{(1)}$ and $\Omega^{(2)}$ as

$$-\int_{\Omega^{(k)}} A^{(k)}\boldsymbol{u}^{(k)} \cdot \boldsymbol{v}^{(k)} \,\mathrm{d}\boldsymbol{x} = \mathcal{E}_{\Omega^{(k)}}(\boldsymbol{u}^{(k)}, \boldsymbol{v}^{(k)}) - \langle \boldsymbol{\sigma}^{(k)}\boldsymbol{n}, \boldsymbol{v}^{(k)}\rangle_{\partial\Omega^{(k)}}$$

for all $\boldsymbol{v}^{(k)} \in H^1(\Omega^{(k)})$, $k = 1, 2$, where $\mathcal{E}_D$ is the bilinear form

$$\mathcal{E}_D(\boldsymbol{u}, \boldsymbol{v}) := \int_D \sigma_{ij} \frac{\partial}{\partial x_j} v_i \,\mathrm{d}\boldsymbol{x}.$$

Here the stress tensor in $\mathcal{E}$ is given by substituting the first element $\boldsymbol{u}$ of $\mathcal{E}_D(\boldsymbol{u}, \boldsymbol{v})$ into the displacement vector in (2.2). Let $\boldsymbol{u}$ satisfy the equilibrium equations (2.1) which read $A^{(k)}\boldsymbol{u}^{(k)} = \boldsymbol{0}$ for $k = 1, 2$. This implies, in particular, that $A^{(k)}\boldsymbol{u}^{(k)} \in L^2(\Omega^{(k)})$. Then we infer from the Green formulae that the stress vectors $\boldsymbol{\sigma}^{(k)}\boldsymbol{n}$ are well defined in $H^{-1/2}(\partial\Omega^{(k)})$. On $\partial\Omega^{(k)} \cap \partial\Omega$ we suppose that $\boldsymbol{\sigma}^{(k)}\boldsymbol{n} = T\boldsymbol{u}^{(k)}$ are L^2-functions. On $\partial\Omega^{(k)} \cap \Gamma'$, the stress vectors $\boldsymbol{\sigma}^{(k)}\boldsymbol{n} = (-1)^k(\sigma_{12}^{(k)}, \sigma_{22}^{(k)})$ are bounded measures over $C_0(\Gamma')$. Since $C_0(\Gamma')$ are dense in $H_0^{1/2}(\Gamma') = H^{1/2}(\Gamma')$, this defines well the duality pairing $\langle \cdot, \cdot \rangle_{\Gamma'}$ between the boundary traces $v_i^{(k)} \in H^{1/2}(\Gamma')$ and the $H^{-1/2}$-distributions $\sigma_{i2}^{(k)}$, $i = 1, 2$. As the result, for $\boldsymbol{u} = \boldsymbol{u}^{(1)}$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(1)}$ in $\Omega^{(1)}$, and $\boldsymbol{u} = \boldsymbol{u}^{(2)}$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(2)}$ in $\Omega^{(2)}$, we arrive at the Green formula for any $\boldsymbol{v} \in H^1(\Omega \setminus \overline{\Gamma'})$

$$\begin{aligned} -\int_{\Omega\setminus\overline{\Gamma'}} A\boldsymbol{u} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = \mathcal{E}_{\Omega\setminus\overline{\Gamma'}}(\boldsymbol{u}, \boldsymbol{v}) - \int_{\partial\Omega} T\boldsymbol{u} \cdot \boldsymbol{v} \,\mathrm{d}S_{\boldsymbol{x}} + \langle \sigma_{12}^{(1)}, v_1^{(1)}\rangle_{\Gamma'} \\ - \langle \sigma_{12}^{(2)}, v_1^{(2)}\rangle_{\Gamma'} + \langle \sigma_{22}^{(1)}, v_2^{(1)}\rangle_{\Gamma'} - \langle \sigma_{22}^{(2)}, v_2^{(2)}\rangle_{\Gamma'}, \end{aligned}$$

where $A\boldsymbol{u} = A^{(1)}\boldsymbol{u}^{(1)}$ in $\Omega^{(1)}$ and $A\boldsymbol{u} = A^{(2)}\boldsymbol{u}^{(2)}$ in $\Omega^{(2)}$. Accounting for $[\sigma_{i2}] = 0$ across Γ', $i = 1, 2$, and the transmission conditions on the joint part of the interface Γ', we obtain the generalized Green formula fulfilled in $\Omega \setminus \overline{\Gamma}$:

$$(3.1) \quad -\int_{\Omega\setminus\overline{\Gamma}} A\boldsymbol{u} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = \mathcal{E}_{\Omega\setminus\overline{\Gamma}}(\boldsymbol{u}, \boldsymbol{v}) - \int_{\partial\Omega} T\boldsymbol{u} \cdot \boldsymbol{v} \,\mathrm{d}S_{\boldsymbol{x}} + \langle \sigma_{12}, [v_1]\rangle_{\Gamma} + \langle \sigma_{22}, [v_2]\rangle_{\Gamma}$$

for any $\boldsymbol{v} \in H^1(\Omega\backslash\overline{\Gamma})$. The brackets $\langle\sigma_{i2}, [v_i]\rangle_\Gamma$ imply the duality pairing between the functions $[v_i] \in H^{1/2}(\Gamma')$ such that $[v_i] = 0$ at $\Gamma' \setminus \overline{\Gamma}$, which form the Lions-Magenes space $H_{00}^{1/2}(\Gamma)$ endowed with the norm

$$\|\xi\|^2_{H_{00}^{1/2}(\Gamma)} := \int_{-l}^{0} |\xi(x_1)|^2 \, \mathrm{d}x_1 + \int_{-l}^{0}\int_{-l}^{0} \frac{|\xi(x_1) - \xi(y_1)|^2}{|x_1 - y_1|^2} \, \mathrm{d}x_1 \, \mathrm{d}y_1 + \int_{-l}^{0} \frac{|\xi(x_1)|^2}{|x_1|} \, \mathrm{d}x_1,$$

and the $H^{-1/2}$-distributions σ_{i2} from its dual space denoted by $H_{00}^{-1/2}(\Gamma)$. For the detailed description of the spaces at a crack see [19].

Using the equilibrium equations in $\Omega \setminus \overline{\Gamma}$ and the boundary conditions on $\partial\Omega$, from (∗) and (3.1) we arrive at the equation for any $\boldsymbol{v} \in H^1(\Omega \setminus \overline{\Gamma})$

$$\mathcal{E}_{\Omega\backslash\overline{\Gamma}}(\boldsymbol{u}, \boldsymbol{v}) + \langle\sigma_{12}, [v_1]\rangle_\Gamma + \langle\sigma_{22}, [v_2]\rangle_\Gamma = \int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{v} \, \mathrm{d}S_{\boldsymbol{x}}. \tag{3.2}$$

The stress tensor $\boldsymbol{\sigma}$ describes the displacement $\boldsymbol{u}$ such that $\boldsymbol{u} = \boldsymbol{u}^{(1)}$ on $\Omega^{(1)}$ and $\boldsymbol{u} = \boldsymbol{u}^{(2)}$ on $\Omega^{(2)}$, by the respective constitutive law (2.2). Consequently, given σ_{12} and σ_{22}, the variational equation (3.2) together with (2.2) determines $\boldsymbol{u} \in H^1(\Omega \setminus \overline{\Gamma})$ uniquely, if we exclude rigid displacements. In the two dimensional case a rigid displacement can be written in the form

$$\boldsymbol{F}(\boldsymbol{x})\boldsymbol{c} = (c_1 + c_0 x_2, c_2 - c_0 x_1)^{\mathrm{T}}$$

with an arbitrary constant vector $\boldsymbol{c} = (c_1, c_2, c_0)^{\mathrm{T}}$. We denote the set of all rigid displacements by $\mathcal{R}$. If we substitute an arbitrary $\boldsymbol{F}(\boldsymbol{x})\boldsymbol{c} \in \mathcal{R}$ as the test function into (3.2), due to $[\boldsymbol{F}(\boldsymbol{x})\boldsymbol{c}] = 0$ and $\nabla\boldsymbol{F}(\boldsymbol{x})\boldsymbol{c} + (\nabla\boldsymbol{F}(\boldsymbol{x})\boldsymbol{c})^{\mathrm{T}} = 0$, we derive the necessary compatibility condition in the usual form

$$\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{F}(\boldsymbol{x})\boldsymbol{c} \, \mathrm{d}S_{\boldsymbol{x}} = 0 \quad \forall\, \boldsymbol{F}(\boldsymbol{x})\boldsymbol{c} \in \mathcal{R}.$$

For admissible stresses at the crack we introduce the dual cone $\mathcal{M} \supset M$ by

$$\mathcal{M} = \{\boldsymbol{p} = (p_1, p_2) \in H_{00}^{-1/2}(\Gamma) \colon\ |\langle p_1, \xi\rangle_\Gamma| \leqslant -\langle f p_2, |\xi|\rangle_\Gamma \ \ \forall\, \xi \in H_{00}^{1/2}(\Gamma)\},$$

which is convex and weakly closed. Note that this set implies also $p_2 \leqslant 0$ in the weak sense, that is, $\langle p_2, \xi\rangle_\Gamma \leqslant 0$ for all $\xi \in H_{00}^{1/2}(\Gamma)$ such that $\xi \geqslant 0$. On $\mathcal{M}$, inequalities (2.10) and (2.11) have the weak form

$$\langle p_2 - \sigma_{22}, [u_2]\rangle_\Gamma \leqslant 0, \tag{3.3}$$

$$\langle p_1 - \sigma_{12}, [u_1]\rangle_\Gamma + \langle f(p_2 - \sigma_{22}), |[u_1]|\rangle_\Gamma \leqslant 0 \quad \forall\, \boldsymbol{p} \in \mathcal{M}. \tag{3.4}$$

Therefore, the *primal-dual weak formulation* of the problem $(*)$ reads: find the displacement $\boldsymbol{u} \in H^1(\Omega \setminus \overline{\Gamma}) \setminus \mathcal{R}$ and the boundary stress $(\sigma_{12}, \sigma_{22}) \in \mathcal{M}$ satisfying the relations (3.2)–(3.4).

Due to the non-penetration condition on Γ we introduce the set of admissible displacements as

$$\mathcal{K} = \{\boldsymbol{v} \in H^1(\Omega \setminus \overline{\Gamma}) \setminus \mathcal{R}\colon\ [v_2] \geqslant 0 \text{ on } \Gamma\}.$$

On the crack Γ the trace theorem guarantees that $[v_1], [v_2] \in H_{00}^{1/2}(\Gamma)$. Relations (3.3) and (3.4) are equivalent to the following complementarity conditions:

$$\begin{gathered}[u_2] \geqslant 0, \quad \langle \sigma_{22}, \xi \rangle_\Gamma \leqslant 0 \quad \forall\, \xi \in H_{00}^{1/2}(\Gamma) \text{ such that } \xi \geqslant 0, \quad \langle \sigma_{22}, [u_2] \rangle_\Gamma = 0, \\ |\langle \sigma_{12}, \xi \rangle_\Gamma| \leqslant -\langle f\sigma_{22}, |\xi| \rangle_\Gamma \quad \forall\, \xi \in H_{00}^{1/2}(\Gamma), \quad \langle \sigma_{12}, [u_1] \rangle_\Gamma + \langle f\sigma_{22}, |[u_1]| \rangle_\Gamma = 0.\end{gathered}$$

Therefore, substituting $\boldsymbol{v} - \boldsymbol{u}$ with $\boldsymbol{v} \in \mathcal{K}$ as the test function into (3.2), we can exclude the dual variables $(\sigma_{12}, \sigma_{22})$ and arrive at the usual quasi-variational inequality: find $\boldsymbol{u} \in \mathcal{K}$ satisfying for an arbitrary $\boldsymbol{v} \in \mathcal{K}$

$$\mathcal{E}_{\Omega\setminus\overline{\Gamma}}(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - \langle f\sigma_{22}, |[v_1]| - |[u_1]| \rangle_\Gamma \geqslant \int_{\partial\Omega} \boldsymbol{g} \cdot (\boldsymbol{v} - \boldsymbol{u}) \, \mathrm{d}S_{\boldsymbol{x}}. \tag{3.5}$$

For smooth $\boldsymbol{u}$, from (3.5) we infer the boundary value problem $(*)$. See [24] for the detailed derivation of the boundary conditions at the crack. We collect the above consideration in the following lemma.

Lemma 3.1. *For a solution pair $\boldsymbol{u} \in H^1(\Omega \setminus \overline{\Gamma}) \setminus \mathcal{R}$ and $(\sigma_{12}, \sigma_{22}) \in \mathcal{M}$ satisfying the primal-dual problem* (3.2)–(3.4), *its primal variable $\boldsymbol{u}$ is in $\mathcal{K}$ and satisfies also* (3.5). *Conversely, for a solution $\boldsymbol{u} \in \mathcal{K}$ of the quasi-variational inequality* (3.5), *the dual variables $(\sigma_{12}, \sigma_{22}) \in \mathcal{M}$ are determined from* (3.2) *and satisfy* (3.3), (3.4).

3.1. The existence theorem

In this subsection we establish the solvability of the quasi-variational inequality (3.5) equivalent to (3.2)–(3.4).

Let us start with some preliminaries. We suppose that the bilinear form in (3.2) satisfies the second Korn inequality: there exist $0 < \underline{C}_0 \leqslant \overline{C}_0 < \infty$ such that

$$\underline{C}_0 \|\boldsymbol{u}\|^2_{H^1(\Omega\setminus\overline{\Gamma})} \leqslant \mathcal{E}_{\Omega\setminus\overline{\Gamma}}(\boldsymbol{u}, \boldsymbol{u}) \leqslant \overline{C}_0 \|\boldsymbol{u}\|^2_{H^1(\Omega\setminus\overline{\Gamma})} \quad \forall\, \boldsymbol{u} \in H^1(\Omega \setminus \overline{\Gamma}) \setminus \mathcal{R}. \tag{3.6}$$

This allows us to introduce the equivalent norm in $H^1(\Omega \setminus \overline{\Gamma}) \setminus \mathcal{R}$ as

$$\|\boldsymbol{u}\|^2_{1,\Omega\setminus\overline{\Gamma}} := \mathcal{E}_{\Omega\setminus\overline{\Gamma}}(\boldsymbol{u}, \boldsymbol{u}).$$

The continuity property of the trace operator on the boundaries of $\Omega^{(k)}$, $k = 1, 2$, implies that there exist C_1, C_2 such that $1 \leqslant C_1 C_2 < \infty$ and

$$\|[\boldsymbol{u}]\|_{H_{00}^{1/2}(\Gamma)} \leqslant C_1 \|\boldsymbol{u}\|_{1,\Omega\setminus\overline{\Gamma}} \quad \forall \boldsymbol{u} \in H^1(\Omega \setminus \overline{\Gamma}) \setminus \mathcal{R}, \tag{3.7}$$

$$\|(\sigma_{12}, \sigma_{22})\|_{H_{00}^{-1/2}(\Gamma)} \leqslant C_2 \|\boldsymbol{u}\|_{1,\Omega\setminus\overline{\Gamma}} \quad \forall \boldsymbol{u} \in H^1(\Omega \setminus \overline{\Gamma}) \setminus \mathcal{R} \text{ such that } A\boldsymbol{u} = \boldsymbol{0}. \tag{3.8}$$

The constants $\underline{C}_0$, $\overline{C}_0$, C_1, C_2 in (3.6)–(3.8) depend on the material parameters $\lambda^{(k)}$, $\mu^{(k)}$ for $k = 1, 2$, and on the geometry of Ω.

To state the existence result we need suitable regularization and penalization. For a small parameter $\varepsilon > 0$, using the infeasible approximation $\sigma_{22}^{\varepsilon} = -(1/\varepsilon)\{[u_2^{\varepsilon}]\}^-$ (compare to (2.13)) we consider the penalized problem: find $\boldsymbol{u}^{\varepsilon} \in H^1(\Omega \setminus \overline{\Gamma}) \setminus \mathcal{R}$ satisfying for an arbitrary $\boldsymbol{v} \in H^1(\Omega \setminus \overline{\Gamma}) \setminus \mathcal{R}$ the variational inequality of the second kind

$$\mathcal{E}_{\Omega\setminus\overline{\Gamma}}(\boldsymbol{u}^{\varepsilon}, \boldsymbol{v} - \boldsymbol{u}^{\varepsilon}) - \int_{\Gamma} \frac{\{[u_2^{\varepsilon}]\}^-}{\varepsilon} [v_2 - u_2^{\varepsilon}] \, \mathrm{d}x_1 + \int_{\Gamma} f \frac{\{[u_2^{\varepsilon}]\}^-}{\varepsilon} (|[v_1]| - |[u_1^{\varepsilon}]|) \, \mathrm{d}x_1 \geqslant \int_{\partial\Omega} \boldsymbol{g} \cdot (\boldsymbol{v} - \boldsymbol{u}^{\varepsilon}) \, \mathrm{d}S_{\boldsymbol{x}}. \tag{3.9}$$

Let $B_{\varrho}(\boldsymbol{O})$ and $B_{\varrho}(\boldsymbol{P})$ be disks of the radius $\varrho > 0$ centered at the crack ends $\boldsymbol{O}$ and $\boldsymbol{P}$, respectively. We introduce a Lipschitz continuous cut-off function η^{ϱ} such that $0 \leqslant \eta^{\varrho}(\boldsymbol{x}) \leqslant 1$, which is supported in $B_{\varrho}(\boldsymbol{O}) \cup B_{\varrho}(\boldsymbol{P})$ and $\eta^{\varrho} = 1$ in $B_{\varrho/2}(\boldsymbol{O}) \cup B_{\varrho/2}(\boldsymbol{P})$. With this notation we formulate the following result.

Lemma 3.2. *For every fixed $\varepsilon > 0$ there exists a solution to problem* (3.9)*. It satisfies the uniform estimate*

$$\|\boldsymbol{u}^{\varepsilon}\|_{1,\Omega\setminus\overline{\Gamma}} + \|\sigma_{12}^{\varepsilon}\|_{H_{00}^{-1/2}(\Gamma)} + \|\sigma_{22}^{\varepsilon}\|_{H_{00}^{-1/2}(\Gamma)} \leqslant c. \tag{3.10}$$

Assume that the friction coefficient is bounded by

$$f < \frac{1}{C_1 C_2} \leqslant 1 \tag{3.11}$$

with C_1, C_2 from (3.7), (3.8)*. Then, for a fixed $\varrho > 0$, the estimate*

$$\|(1 - \eta^{\varrho})\sigma_{22}^{\varepsilon}\|_{H^{-1/2+\tau}(\Gamma)} \leqslant \mathrm{C}(\varrho), \quad \tau \in (0, 1/2], \tag{3.12}$$

holds and is uniform with respect to ε but not ϱ.

Indeed, using a proper regularization of the non-differentiable term in (3.9) the existence of a solution can be stated for all data. Also, its local smoothness inside the contact boundary for the friction coefficient sufficiently small was shown in many works. To this end we refer to [1], [8], [24], [28].

The principal difficulty concerns the fact that the additional smoothness stated in (3.12) is not preserved when $\varrho \to 0$. We state an auxiliary result associated with the Saint-Venant principle in the following lemma.

Lemma 3.3. *There exist* $\varrho_0 > 0$ *and* $0 < \alpha < \infty$ *such that the estimate*

$$\|\boldsymbol{u}^\varepsilon\|_{1,B_\varrho(\boldsymbol{O})\setminus\overline{\Gamma}} \leqslant \Big(\frac{\varrho}{\varrho_0}\Big)^{1/\alpha} \|\boldsymbol{u}^\varepsilon\|_{1,B_{\varrho_0}(\boldsymbol{O})\setminus\overline{\Gamma}} \quad \forall\,\varrho \in (0,\varrho_0] \tag{3.13}$$

holds for the solution $\boldsymbol{u}^\varepsilon \in H^1(\Omega\setminus\overline{\Gamma})\setminus\mathcal{R}$ *of the penalized inequality* (3.9). *In the neighbourhood* $B_{\varrho_0}(\boldsymbol{P})$ *of the crack end* $\boldsymbol{P} \in \partial\Omega$, *let the boundaries* $\partial\Omega^{(k)}$, $k = 1, 2$, *be locally straight lines and* $\boldsymbol{g} = \boldsymbol{0}$. *Then there exists* $0 < \alpha_1 < \infty$ *such that*

$$\|\boldsymbol{u}^\varepsilon\|_{1,(B_\varrho(\boldsymbol{P})\cap\Omega)\setminus\overline{\Gamma}} \leqslant \Big(\frac{\varrho}{\varrho_0}\Big)^{1/\alpha_1} \|\boldsymbol{u}^\varepsilon\|_{1,(B_{\varrho_0}(\boldsymbol{P})\cap\Omega)\setminus\overline{\Gamma}} \quad \forall\,\varrho \in (0,\varrho_0]. \tag{3.14}$$

P r o o f. Let us consider the solution $\boldsymbol{u}^\varepsilon$ of (3.9). We focus on the crack tip $\boldsymbol{O}$ and after that we modify the arguments for $\boldsymbol{P}$.

For fixed $\varrho > 0$ such that $B_\varrho(\boldsymbol{O}) \subset \Omega$, similarly to (3.1) the Green formula yields

$$\begin{aligned}\mathcal{E}_{B_\varrho(\boldsymbol{O})\setminus\overline{\Gamma}}(\boldsymbol{u}^\varepsilon,\boldsymbol{v}) - \int_{\Gamma\cap B_\varrho(\boldsymbol{O})} \frac{\{[u_2^\varepsilon]\}^-}{\varepsilon}[v_2]\,\mathrm{d}x_1 + \langle\sigma_{12}^\varepsilon,[v_1]\rangle_{\Gamma\cap B_\varrho(\boldsymbol{O})}\\ = \int_{\partial B_\varrho(\boldsymbol{O})} T\boldsymbol{u}^\varepsilon\cdot\boldsymbol{v}\,\mathrm{d}S_{\boldsymbol{x}} \quad \forall\,\boldsymbol{v}\in H^1(B_\varrho(\boldsymbol{O})\setminus\overline{\Gamma}).\end{aligned} \tag{3.15}$$

From (3.9) we infer the following boundary conditions at the crack Γ:

$$-\frac{\{[u_2^\varepsilon]\}^-}{\varepsilon}[u_2^\varepsilon] = \frac{(\{[u_2^\varepsilon]\}^-)^2}{\varepsilon} \geqslant 0, \quad \sigma_{12}^\varepsilon[u_1^\varepsilon] = f\frac{\{[u_2^\varepsilon]\}^-}{\varepsilon}|[u_1^\varepsilon]| \geqslant 0.$$

Therefore, the substitution of $\boldsymbol{v} = \boldsymbol{u}^\varepsilon$ into (3.15) results in the inequality

$$\mathcal{E}_{B_\varrho(\boldsymbol{O})\setminus\overline{\Gamma}}(\boldsymbol{u}^\varepsilon,\boldsymbol{u}^\varepsilon) \leqslant \int_{\partial B_\varrho(\boldsymbol{O})} T\boldsymbol{u}^\varepsilon\cdot\boldsymbol{u}^\varepsilon\,\mathrm{d}S_{\boldsymbol{x}}. \tag{3.16}$$

In the neighbourhood $B_\varrho(\boldsymbol{O})\setminus\overline{\Gamma} = \{x_1 = r\cos\theta,\ x_2 = r\sin\theta\colon\ r \leqslant \varrho,\ \theta\in(-\pi,\pi)\}$ we decompose $\boldsymbol{u}^\varepsilon$ into the rigid displacement $\boldsymbol{F}(\boldsymbol{x})\boldsymbol{c}^\varepsilon \in \mathcal{R}$ with $\boldsymbol{c}^\varepsilon := (c_1^\varepsilon, c_2^\varepsilon, c_0^\varepsilon)^{\mathrm{T}}$ and $\boldsymbol{U}^\varepsilon \in H^1(B_\varrho(\boldsymbol{O})\setminus\overline{\Gamma})\setminus\mathcal{R}$ such that

$$\begin{gathered}(c_1^\varepsilon, c_2^\varepsilon)^{\mathrm{T}} := \frac{1}{\pi\varrho^2}\int_{B_\varrho(\boldsymbol{O})} \boldsymbol{u}^\varepsilon\,\mathrm{d}\boldsymbol{x}, \quad c_0^\varepsilon := \frac{2}{\pi\varrho^4}\int_{B_\varrho(\boldsymbol{O})} (u_1^\varepsilon x_2 - u_2^\varepsilon x_1)\,\mathrm{d}\boldsymbol{x},\\ \boldsymbol{U}^\varepsilon(\boldsymbol{x}) := \boldsymbol{u}^\varepsilon(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{x})\boldsymbol{c}^\varepsilon.\end{gathered} \tag{3.17}$$

Indeed, from (3.17) we easily derive

$$\int_{B_\varrho(\boldsymbol{O})} \boldsymbol{U}^\varepsilon \,\mathrm{d}\boldsymbol{x} = \boldsymbol{0}, \quad \int_{B_\varrho(\boldsymbol{O})} (U_1^\varepsilon x_2 - U_2^\varepsilon x_1)\,\mathrm{d}\boldsymbol{x} = 0,$$

which implies the orthogonal decomposition in the sense that

$$\int_{B_\varrho(\boldsymbol{O})} \boldsymbol{U}^\varepsilon \cdot \boldsymbol{F}(\boldsymbol{x})\boldsymbol{c}\,\mathrm{d}\boldsymbol{x} = 0 \quad \forall\, \boldsymbol{F}(\boldsymbol{x})\boldsymbol{c} \in \mathcal{R}. \tag{3.18}$$

Taking an arbitrary rigid displacement $\boldsymbol{F}(\boldsymbol{x})\boldsymbol{c} \in \mathcal{R}$ as the test function in (3.15), due to $[\boldsymbol{F}(\boldsymbol{x})\boldsymbol{c}] = 0$ and $\nabla \boldsymbol{F}(\boldsymbol{x})\boldsymbol{c} + (\nabla \boldsymbol{F}(\boldsymbol{x})\boldsymbol{c})^{\mathrm{T}} = 0$ we obtain the equalities

$$\int_{\partial B_\varrho(\boldsymbol{O})} T\boldsymbol{u}^\varepsilon \,\mathrm{d}S_{\boldsymbol{x}} = \boldsymbol{0}, \quad \int_{\partial B_\varrho(\boldsymbol{O})} T\boldsymbol{u}^\varepsilon \cdot (x_2, -x_1)\,\mathrm{d}S_{\boldsymbol{x}} = 0.$$

Therefore, the substitution of decomposition (3.17) into the boundary integral in (3.16) provides

$$\mathcal{E}_{B_\varrho(\boldsymbol{O})\setminus\overline{\Gamma}}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) \leqslant \int_{\partial B_\varrho(\boldsymbol{O})} T\boldsymbol{u}^\varepsilon \cdot \boldsymbol{U}^\varepsilon \,\mathrm{d}S_{\boldsymbol{x}}. \tag{3.19}$$

From (2.2) we calculate the upper bound $C_3 > 0$ such that $\sigma_{ij}\sigma_{ij} \leqslant C_3\sigma_{ij}(\partial/\partial x_j)u_i$ for all $\boldsymbol{u}$, and estimate the right-hand side of (3.19) as

$$\int_{\partial B_\varrho(\boldsymbol{O})} T\boldsymbol{u}^\varepsilon \cdot \boldsymbol{U}^\varepsilon \,\mathrm{d}S_{\boldsymbol{x}} \leqslant \frac{C_3\varrho}{\alpha_2}\mathcal{E}_{\partial B_\varrho(\boldsymbol{O})}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) + \frac{\alpha_2}{4\varrho}\int_{\partial B_\varrho(\boldsymbol{O})} |\boldsymbol{U}^\varepsilon|^2 \,\mathrm{d}S_{\boldsymbol{x}} \tag{3.20}$$

for arbitrary $\alpha_2 > 0$. If $\boldsymbol{U}^\varepsilon \equiv 0$ on $\partial B_\varrho(\boldsymbol{O})$, then (3.19) and (3.20) immediately imply the desired estimate (3.21) with $\frac{1}{2}\alpha^{-1} = \alpha_2/C_3$. Otherwise, to prove (3.21) for $\boldsymbol{U}^\varepsilon \not\equiv 0$ on $\partial B_\varrho(\boldsymbol{O})$ we evaluate $|\boldsymbol{U}^\varepsilon|^2$ on the circle with help of the Rayleigh principle, see [32]. For this reason, let us define the non-negative functional

$$J(\varrho, \boldsymbol{U}) := \mathcal{E}_{B_\varrho(\boldsymbol{O})\setminus\overline{\Gamma}}(\boldsymbol{U}, \boldsymbol{U})\Big(\int_{\partial B_\varrho(\boldsymbol{O})} |\boldsymbol{U}|^2 \,\mathrm{d}S_{\boldsymbol{x}}\Big)^{-1}.$$

If $J(\varrho, \boldsymbol{U}^\varepsilon)$ vanishes, it means exactly $\boldsymbol{U}^\varepsilon \in \mathcal{R}$, which contradicts (3.18). We claim that $J(\varrho, \mathbf{U}^\varepsilon) > 0$, and estimate it from below. In fact, by virtue of the second Korn inequality and the uniform continuity of the trace operator on the boundary, there exists $J(\varrho) > 0$ such that

$$J(\varrho) = \min J(\varrho, \boldsymbol{U}) \text{ over all } \boldsymbol{U} \in H^1(B_\varrho(\boldsymbol{O}) \setminus \overline{\Gamma}) \setminus \mathcal{R}, \quad \boldsymbol{U} \not\equiv 0 \text{ on } \partial B_\varrho(\boldsymbol{O}).$$

Next we apply the homogeneity argument. The coordinate change $\boldsymbol{y} = \boldsymbol{x}/\varrho$ transforms $B_\varrho(\boldsymbol{O})$ onto $B_1(\boldsymbol{O})$, and $J(\varrho, \boldsymbol{U}(\boldsymbol{x})) = \varrho^{-1} J(1, \boldsymbol{U}(\varrho \boldsymbol{x}))$. In $B_1(\boldsymbol{O})$ we have

$$J(1) = \min J(1, \boldsymbol{U}) \text{ over all } \boldsymbol{U} \in H^1(B_1(\boldsymbol{O}) \setminus \overline{\Gamma}) \setminus \mathcal{R}, \quad \boldsymbol{U} \not\equiv 0 \text{ on } \partial B_1(\boldsymbol{O}),$$

and $J(1) > 0$ by the above argument. Henceforth, $J(\varrho) = \varrho^{-1} J(1)$, and we have

$$\int_{\partial B_\varrho(\boldsymbol{O})} |\boldsymbol{U}^\varepsilon|^2 \, \mathrm{d}S_{\boldsymbol{x}} \leqslant \frac{1}{J(\varrho)} \mathcal{E}_{B_\varrho(\boldsymbol{O}) \setminus \overline{\Gamma}}(\boldsymbol{U}^\varepsilon, \boldsymbol{U}^\varepsilon) = \frac{\varrho}{J(1)} \mathcal{E}_{B_\varrho(\boldsymbol{O}) \setminus \overline{\Gamma}}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon)$$

due to (3.17). Substituting this into (3.20) and taking $0 < \alpha_2 < 4J(1)$, from (3.19) we derive the estimate with $\frac{1}{2}\alpha^{-1} = (\alpha_2/C_3)(1 - (\alpha_2/4J(1))$:

$$\text{(3.21)} \qquad \frac{1}{2\alpha} \mathcal{E}_{B_\varrho(\boldsymbol{O}) \setminus \overline{\Gamma}}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) \leqslant \varrho \mathcal{E}_{\partial B_\varrho(\boldsymbol{O})}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon).$$

Finally, applying the Reynolds transport theorem

$$\mathcal{E}_{\partial B_\varrho(\boldsymbol{O})}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) = \frac{\mathrm{d}}{\mathrm{d}\varrho} \mathcal{E}_{B_\varrho(\boldsymbol{O}) \setminus \overline{\Gamma}}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon),$$

and the Grönwall lemma, (3.21) results in the assertion (3.13).

In the circular sector $B_\varrho(\boldsymbol{P}) \cap \Omega$ bounded by the line segments of $\partial\Omega \cap \partial\Omega^{(1)}$ and $\partial\Omega \cap \partial\Omega^{(2)}$ around the crack end $\boldsymbol{P} \in \partial\Omega$, when $\boldsymbol{g} = \boldsymbol{0}$ we can repeat the above argument for $\boldsymbol{u}^\varepsilon \in H^1((B_\varrho(\boldsymbol{P}) \cap \Omega) \setminus \overline{\Gamma})$ and thus obtain (3.14). □

Let us apply Lemma 3.3 to the specific case when $\sigma_{ij}^\varepsilon = (\partial/\partial x_j) u_i^\varepsilon = (\partial/\partial x_i) u_j^\varepsilon$. In this case, the exponent $\alpha^{-1} = \frac{1}{2}$ can be calculated exactly, and α_1^{-1} depends on the angle forming around $\boldsymbol{P}$, which is provided by the Wirtinger inequality.

Moreover, we see that the assumption of the straight boundary near $\boldsymbol{P}$ can be avoided, and also the condition $\boldsymbol{g} = \boldsymbol{0}$ can be replaced by $\boldsymbol{g} \cdot \boldsymbol{u}^\varepsilon \leqslant 0$.

Theorem 3.1. *Under the assumptions of Lemma* 3.2 *and Lemma* 3.3, *there exists a solution* $\boldsymbol{u} \in \mathcal{K}$ *of the quasi-variational inequality* (3.5).

P r o o f. Consider the sequence $\{\boldsymbol{u}^\varepsilon\} \in H^1(\Omega \setminus \overline{\Gamma}) \setminus \mathcal{R}$ of solutions of the penalized inequality (3.9). We start with the standard arguments.

As $\varepsilon \to 0$, due to (3.10) and (3.12) with fixed $\varrho > 0$ we can extract a convergent subsequence still denoted by ε such that

$$\text{(3.22)} \qquad \boldsymbol{u}^\varepsilon \to \boldsymbol{u} \text{ weakly in } H^1(\Omega \setminus \overline{\Gamma}), \quad \boldsymbol{\sigma}^\varepsilon \to \boldsymbol{\sigma} \text{ weakly in } L_2(\Omega \setminus \overline{\Gamma}),$$

$$\text{(3.23)} \qquad [\boldsymbol{u}^\varepsilon] \to [\boldsymbol{u}], \quad |[u_1^\varepsilon]| \to |[u_1]| \text{ weakly in } H_{00}^{1/2}(\Gamma),$$

$$\text{(3.24)} \qquad -\frac{1}{\varepsilon}\{[u_2^\varepsilon]\}^- = \sigma_{22}^\varepsilon \to \sigma_{22}, \quad \sigma_{12}^\varepsilon \to \sigma_{12} \text{ weakly in } H_{00}^{-1/2}(\Gamma),$$

$$\text{(3.25)} \qquad (1 - \eta^\varrho)\sigma_{22}^\varepsilon \to (1 - \eta^\varrho)\sigma_{22} \text{ strongly in } H_{00}^{-1/2}(\Gamma).$$

It follows from (3.24) that $\{[u_2^\varepsilon]\}^- \to 0$ as $\varepsilon \to 0$, thus $\boldsymbol{u} \in \mathcal{K}$. The substitution of $\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{v} = 2\boldsymbol{u}^\varepsilon$ into (3.9) gives the equality

$$(3.26)\quad \mathcal{E}_{\Omega\setminus\overline{\Gamma}}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) + \int_\Gamma \frac{(\{[u_2^\varepsilon]\}^-)^2}{\varepsilon}\,\mathrm{d}x_1 + \int_\Gamma f\frac{\{[u_2^\varepsilon]\}^-}{\varepsilon}|[u_1^\varepsilon]|\,\mathrm{d}x_1 = \int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{u}^\varepsilon\,\mathrm{d}S_{\boldsymbol{x}},$$

and the respective inequality for all $\boldsymbol{v} \in H^1(\Omega\setminus\overline{\Gamma})\setminus\mathcal{R}$

$$\mathcal{E}_{\Omega\setminus\overline{\Gamma}}(\boldsymbol{u}^\varepsilon, \boldsymbol{v}) - \int_\Gamma \frac{\{[u_2^\varepsilon]\}^-}{\varepsilon}[v_2]\,\mathrm{d}x_1 + \int_\Gamma f\frac{\{[u_2^\varepsilon]\}^-}{\varepsilon}|[v_1]|\,\mathrm{d}x_1 \geqslant \int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}S_{\boldsymbol{x}}.$$

For $[v_2] \geqslant 0$ it turns into

$$(3.27)\qquad \mathcal{E}_{\Omega\setminus\overline{\Gamma}}(\boldsymbol{u}^\varepsilon, \boldsymbol{v}) + \int_\Gamma f\frac{\{[u_2^\varepsilon]\}^-}{\varepsilon}|[v_1]|\,\mathrm{d}x_1 \geqslant \int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}S_{\boldsymbol{x}} \quad \forall\,\boldsymbol{v}\in\mathcal{K}.$$

Passing in (3.27) to the limit as $\varepsilon \to 0$, in view of (3.22)–(3.24) we obtain

$$(3.28)\qquad \mathcal{E}_{\Omega\setminus\overline{\Gamma}}(\boldsymbol{u}, \boldsymbol{v}) + \langle f\sigma_{22}, |[v_1]|\rangle_\Gamma \geqslant \int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}S_{\boldsymbol{x}} \quad \forall\,\boldsymbol{v}\in\mathcal{K}.$$

The Green formula and (3.28) yield $A\boldsymbol{u} = \boldsymbol{0}$.

The main part is to pass to the limit in equality (3.26). Here we follow the scheme of [1], [18]. While the first quadratic term $\mathcal{E}_{\Omega\setminus\overline{\Gamma}}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon)$ is weakly lower semicontinuous (w.l.s.c.), the principal difficulty concerns the term $-\int_\Gamma f\sigma_{22}^\varepsilon|[u_1^\varepsilon]|\,\mathrm{d}x_1$. To establish its w.l.s.c. property we apply the result of Lemma 3.3. For this reason, we take a monotone sequence of the cut-off functions η^δ such that

$$\eta^\delta(\boldsymbol{x}) \searrow 0, \quad (1-\eta^\delta(\boldsymbol{x})) \nearrow 1 \quad \text{in } L^p(\Gamma) \text{ for } p\in[1,\infty) \text{ as } \delta\to 0,$$

and using (3.25) we derive the consequent estimates (recall that $\sigma_{22}^\varepsilon \leqslant 0$):

$$\begin{aligned}(3.29)\quad -\liminf_{\varepsilon\to 0}\int_\Gamma f\sigma_{22}^\varepsilon|[u_1^\varepsilon]|\,\mathrm{d}x_1 &\geqslant -\liminf_{\varepsilon\to 0}\int_\Gamma f(1-\eta^\delta)\sigma_{22}^\varepsilon|[u_1^\varepsilon]|\,\mathrm{d}x_1\\ &= -\langle f(1-\eta^\delta)\sigma_{22}, |[u_1]|\rangle_\Gamma\\ &\geqslant -\langle f\sigma_{22}, |[u_1]|\rangle_\Gamma + \langle f\sigma_{22}, |[u_1]|\rangle_{\Gamma\cap B_\varrho(\boldsymbol{O})} + \langle f\sigma_{22}, |[u_1]|\rangle_{\Gamma\cap B_\varrho(\boldsymbol{P})}.\end{aligned}$$

Next we apply the estimation like (3.7) and (3.8) to $B_\varrho(\boldsymbol{O})\setminus\overline{\Gamma}$ and $(\Omega\cap B_\varrho(\boldsymbol{P}))\setminus\overline{\Gamma}$. Indeed, following (3.17) we can decompose $\boldsymbol{u} = \boldsymbol{F}(\boldsymbol{x})\boldsymbol{c} + \boldsymbol{U}$ in $B_\varrho(\boldsymbol{O})\setminus\overline{\Gamma}$, and similarly in $(\Omega\cap B_\varrho(\boldsymbol{P}))\setminus\overline{\Gamma}$. Excluding the rigid displacement due to $\boldsymbol{\sigma}(\boldsymbol{F}(\boldsymbol{x})\boldsymbol{c}) = \boldsymbol{0}$ and $[\boldsymbol{F}(\boldsymbol{x})\boldsymbol{c}] = \boldsymbol{0}$ we have

$$\begin{aligned}|\langle\sigma_{22}, |[u_1]|\rangle_{\Gamma\cap B_\varrho(\boldsymbol{O})}| &= |\langle\sigma_{22}(\boldsymbol{U}), |[U_1]|\rangle_{\Gamma\cap B_\varrho(\boldsymbol{O})}| \leqslant \widetilde{C}_1\widetilde{C}_2\|\boldsymbol{u}\|^2_{1,B_\varrho(\boldsymbol{O})\setminus\overline{\Gamma}},\\ |\langle\sigma_{22}, |[u_1]|\rangle_{\Gamma\cap B_\varrho(\boldsymbol{P})}| &= |\langle\sigma_{22}(\boldsymbol{U}), |[U_1]|\rangle_{\Gamma\cap B_\varrho(\boldsymbol{P})}| \leqslant \widetilde{C}_1\widetilde{C}_2\|\boldsymbol{u}\|^2_{1,(B_\varrho(\boldsymbol{P})\cap\Omega)\setminus\overline{\Gamma}}.\end{aligned}$$

The homogeneity argument allows us to choose $\varrho_0 > 0$ such that the upper bounds $\widetilde{C}_1$ and $\widetilde{C}_2$ are uniform with respect to all $\varrho \leqslant \varrho_0$. Henceforth, from Lemma 3.3, (3.10), and w.l.s.c. of the norm we infer

$$\|\boldsymbol{u}\|_{1,B_\varrho(\boldsymbol{O})\setminus\overline{\Gamma}} \leqslant \liminf_{\varepsilon\to 0} \|\boldsymbol{u}^\varepsilon\|_{1,B_\varrho(\boldsymbol{O})\setminus\overline{\Gamma}} \leqslant \liminf_{\varepsilon\to 0} \Big\{ \Big(\frac{\varrho}{\varrho_0}\Big)^{1/\alpha} \|\boldsymbol{u}^\varepsilon\|_{1,\Omega\setminus\overline{\Gamma}} \Big\} \leqslant \varrho^{1/\alpha} C_4,$$
$$\|\boldsymbol{u}\|_{1,(B_\varrho(\boldsymbol{P})\cap\Omega)\setminus\overline{\Gamma}} \leqslant \liminf_{\varepsilon\to 0} \|\boldsymbol{u}^\varepsilon\|_{1,(B_\varrho(\boldsymbol{P})\cap\Omega)\setminus\overline{\Gamma}} \leqslant \varrho^{1/\alpha_1} C_4.$$

By applying the above estimates to (3.29) and by introducing $C_5 = \widetilde{C}_1\widetilde{C}_2 C_4^2$, we arrive at

$$-\liminf_{\varepsilon\to 0} \int_\Gamma f\sigma_{22}^\varepsilon |[u_1^\varepsilon]| \,\mathrm{d}x_1 \geqslant -\langle f\sigma_{22}, |[u_1]|\rangle_\Gamma - (\varrho^{2/\alpha} + \varrho^{2/\alpha_1}) f C_5.$$

By passing $\varrho \to 0$ and using the w.l.s.c. property we obtain

$$-\liminf_{\varepsilon\to 0} \int_\Gamma f\sigma_{22}^\varepsilon |[u_1^\varepsilon]| \,\mathrm{d}x_1 \geqslant -\langle f\sigma_{22}, |[u_1]|\rangle_\Gamma. \tag{3.30}$$

Now we apply (3.30) and pass $-\int_\Gamma \sigma_{22}^\varepsilon [u_2^\varepsilon] \,\mathrm{d}x_1$ to the limit. We rewrite (3.26) as

$$\int_\Gamma \sigma_{22}^\varepsilon [u_2^\varepsilon] \,\mathrm{d}x_1 = \mathcal{E}_{\Omega\setminus\overline{\Gamma}}(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) - \int_\Gamma f\sigma_{22}^\varepsilon |[u_1^\varepsilon]| \,\mathrm{d}x_1 - \int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{u}^\varepsilon \,\mathrm{d}S_{\boldsymbol{x}}.$$

Due to the w.l.s.c. property, for $\varepsilon \to 0$ this results in the limit

$$0 \geqslant \liminf_{\varepsilon\to 0} \int_\Gamma \sigma_{22}^\varepsilon [u_2^\varepsilon] \,\mathrm{d}x_1 \geqslant \mathcal{E}_{\Omega\setminus\overline{\Gamma}}(\boldsymbol{u}, \boldsymbol{u}) - \langle f\sigma_{22}, |[u_1]|\rangle_\Gamma - \int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{u} \,\mathrm{d}S_{\boldsymbol{x}}. \tag{3.31}$$

On the other hand, substituting $\boldsymbol{v} = \boldsymbol{u}$ into (3.27) yields the converse inequality

$$\mathcal{E}_{\Omega\setminus\overline{\Gamma}}(\boldsymbol{u}, \boldsymbol{u}) - \langle f\sigma_{22}, |[u_1]|\rangle_\Gamma - \int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{u} \,\mathrm{d}S_{\boldsymbol{x}} \geqslant 0. \tag{3.32}$$

Henceforth, from (3.31) and (3.32) we arrive at the equality

$$\mathcal{E}_{\Omega\setminus\overline{\Gamma}}(\boldsymbol{u}, \boldsymbol{u}) - \langle f\sigma_{22}, |[u_1]|\rangle_\Gamma = \int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{u} \,\mathrm{d}S_{\boldsymbol{x}}. \tag{3.33}$$

Consequently, (3.28) and (3.33) are exactly the quasi-variational inequality (3.5). This completes the proof. □

We note that the existence theorem can be extended with a non-constant friction coefficient $f \in L^\infty(\Gamma)$, $f \geqslant 0$, which should be a multiplier on $H_{00}^{1/2}(\Gamma)$ and satisfy (3.11) in the respective norms.

For the need of further asymptotic analysis in Section 4 we formulate the following lemma on the local smoothness of the solution.

Lemma 3.4. *The solution $\boldsymbol{u} \in \mathcal{K}$ of the quasi-variational inequality* (3.5) *obeys the interior C^∞-regularity on $\Omega^{(1)}$ and $\Omega^{(2)}$. The boundary stress components σ_{i2}, $i = 1, 2$ are pointwise functions inside the crack Γ.*

Indeed, the interior C^∞-regularity of $\boldsymbol{u}$ is ensured by the equilibrium equation $A\boldsymbol{u} = \mathbf{0}$ in the standard way (e.g., [10]). The interior regularity at the crack follows from Lemma 3.2. For more results concerning regularity of the solution due to the frictional crack see [3], [24].

4. Convergent expansions of the solution near the crack tip

In this section we derive convergent expansions of the solution constructed in Theorem 3.1. For this purpose we assume that on the whole crack $B_\varrho(\mathbf{0}) \cap \Gamma$ one of three cases mentioned in Section 2 occurs: open crack, stick state, slip state, that is, there are no switches among the three cases on $B_\varrho(\boldsymbol{O}) \cap \Gamma$.

Now we introduce a polar coordinate system $(x_1, x_2) = (r\cos\theta, r\sin\theta)$ with respect to the origin $\boldsymbol{O}$. And we fix some notation:

$$B_\varrho := B_\varrho(\boldsymbol{O}), \quad B_\varrho^{(1)} := B_\varrho \cap \Omega^{(1)}, \quad B_\varrho^{(2)} := B_\varrho \cap \Omega^{(2)}$$

with a sufficiently small ϱ such that $B_\varrho^{(k)} \subset \Omega^{(k)}$ $(k = 1, 2)$.

Next, we construct the Goursat-Kolosov-Muskhelishvili stress functions, see [27], in each $B_\varrho^{(k)}$. The interior and boundary regularity results of Lemma 3.4 ensure that $\sigma_{ij}^{(k)}$ is in $C^\infty(B_\varrho^{(k)})$ and satisfies the conditions on the crack in the pointwise sense. From this fact and the Poincaré lemma we obtain two holomorphic functions $\varphi^{(k)}(z)$, $\omega^{(k)}(z)$ in $B_\varrho^{(k)}$ $(k = 1, 2)$ of the complex variable $z = x_1 + \mathrm{i}x_2$. Moreover, it follows from the generalized Poincaré lemma (e.g., [15], [16]) that $\varphi^{(k)}(z)$, $\omega^{(k)}(z) \in H^1(B_\varrho^{(k)})$. Then for each $k = 1, 2$ the displacement $\boldsymbol{u}^{(k)}$ and the stress fields $\boldsymbol{\sigma}^{(k)}$ in the plane isotropic elasticity $B_\varrho^{(k)}$ can be represented as

$$2\mu^{(k)}(u_1^{(k)} + \mathrm{i}u_2^{(k)}) = \tilde{\kappa}^{(k)}\varphi^{(k)}(z) - \overline{\omega^{(k)}(z)} + (\overline{z} - z)\overline{\varphi^{(k)\prime}(z)}, \tag{4.1}$$

$$\sigma_{11}^{(k)} + \sigma_{22}^{(k)} = 2(\varphi^{(k)\prime}(z) + \overline{\varphi^{(k)\prime}(z)}), \tag{4.2}$$

$$\sigma_{22}^{(k)} - \mathrm{i}\sigma_{12}^{(k)} = \varphi^{(k)\prime}(z) + \overline{\omega^{(k)\prime}(z)} + (z - \overline{z})\overline{\varphi^{(k)\prime\prime}(z)}, \tag{4.3}$$

where $\varphi^{(k)\prime}(z) = \mathrm{d}\varphi^{(k)}/\mathrm{d}z$ and a bar over a function denotes the complex conjugate.

4.1. Case 1 (open crack)

In this case the condition (2.3) means a traction-free condition on the crack. Hence, following [30] and [9] we consider the behaviour of the stress functions near the crack tip. It follows from the problem (∗) and (2.3) that

$$[\sigma_{22} - \mathrm{i}\sigma_{12}] = 0 \quad \text{on} \quad B_\varrho \cap \Gamma'.$$

Since $z = \overline{z}$ on Γ', from (4.3) we have

$$\varphi^{(1)'}(x_1) + \overline{\omega^{(1)'}(x_1)} = \varphi^{(2)'}(x_1) + \overline{\omega^{(2)'}(x_1)} \quad \text{on } B_\varrho \cap \Gamma'.$$

Here it is easy to see that $\overline{\omega^{(1)'}(\overline{z})}$ and $\overline{\omega^{(2)'}(\overline{z})}$ are holomorphic in $B_\varrho^{(2)}$ and $B_\varrho^{(1)}$, respectively. Therefore, we know that

$$\varphi^{(1)'}(z) - \overline{\omega^{(2)'}(\overline{z})} = \varphi^{(2)'}(z) - \overline{\omega^{(1)'}(\overline{z})} \quad \text{on } B_\varrho \cap \Gamma'.$$

Then we can define a holomorphic function $\Phi(z)$ on the whole B_ϱ as

$$2\Phi(z) = \begin{cases} \varphi^{(1)'}(z) - \overline{\omega^{(2)'}(\overline{z})} & \text{in } B_\varrho^{(1)}, \\ \varphi^{(2)'}(z) - \overline{\omega^{(1)'}(\overline{z})} & \text{in } B_\varrho^{(2)}. \end{cases}$$

On the bonded part $B_\varrho \cap (\Gamma' \setminus \overline{\Gamma})$, by the condition $[\boldsymbol{u}] = \mathbf{0}$ and (4.1) we obtain

$$\frac{\tilde{\kappa}^{(1)}}{\mu^{(1)}}\varphi^{(1)}(x_1) - \frac{1}{\mu^{(1)}}\overline{\omega^{(1)}(x_1)} = \frac{\tilde{\kappa}^{(2)}}{\mu^{(2)}}\varphi^{(2)}(x_1) - \frac{1}{\mu^{(2)}}\overline{\omega^{(2)}(x_1)} \quad \text{on } B_\varrho \cap (\Gamma' \setminus \overline{\Gamma}).$$

Differentiating both sides of this equality with respect to x_1 yields

$$\frac{\tilde{\kappa}^{(1)}}{\mu^{(1)}}\varphi^{(1)'}(x_1) - \frac{1}{\mu^{(1)}}\overline{\omega^{(1)'}(x_1)} = \frac{\tilde{\kappa}^{(2)}}{\mu^{(2)}}\varphi^{(2)'}(x_1) - \frac{1}{\mu^{(2)}}\overline{\omega^{(2)'}(x_1)} \quad \text{on } B_\varrho \cap (\Gamma' \setminus \overline{\Gamma}).$$

Hence we can define a sectionally holomorphic function in B_ϱ cut along $B_\varrho \cap \Gamma$, i.e. holomorphic in $B_\varrho \setminus \overline{\Gamma}$, sectionally continuous in the neighbourhood of $B_\varrho \cap \Gamma$, weakly singular at the end points ($z = 0$, $z = -\varrho$),

$$\Psi(z) = \begin{cases} \dfrac{\tilde{\kappa}^{(1)}}{\mu^{(1)}}\varphi^{(1)'}(z) + \dfrac{1}{\mu^{(2)}}\overline{\omega^{(2)'}(\overline{z})} & \text{in } B_\varrho^{(1)}, \\ \dfrac{\tilde{\kappa}^{(2)}}{\mu^{(2)}}\varphi^{(2)'}(z) + \dfrac{1}{\mu^{(1)}}\overline{\omega^{(1)'}(\overline{z})} & \text{in } B_\varrho^{(2)}. \end{cases}$$

Next, by using functions $\Phi(z)$, $\Psi(z)$ we express the functions $\varphi^{(k)}(z)$, $\omega^{(k)}(z)$ $(k=1,2)$ as

$$\varphi^{(1)'}(z) = \frac{1}{m_1}\Big(\Psi(z) + \frac{2}{\mu^{(2)}}\Phi(z)\Big) \quad \text{in } B_\varrho^{(1)}, \tag{4.4}$$

$$\overline{\omega^{(2)'}(\overline{z})} = \frac{1}{m_1}\Big(\Psi(z) - \frac{2\tilde{\kappa}^{(1)}}{\mu^{(1)}}\Phi(z)\Big) \quad \text{in } B_\varrho^{(1)}, \tag{4.5}$$

$$\varphi^{(2)'}(z) = \frac{1}{m_2}\Big(\Psi(z) + \frac{2}{\mu^{(1)}}\Phi(z)\Big) \quad \text{in } B_\varrho^{(2)}, \tag{4.6}$$

$$\overline{\omega^{(1)'}(\overline{z})} = \frac{1}{m_2}\Big(\Psi(z) - \frac{2\tilde{\kappa}^{(2)}}{\mu^{(2)}}\Phi(z)\Big) \quad \text{in } B_\varrho^{(2)}, \tag{4.7}$$

where

$$m_1 := \frac{\tilde{\kappa}^{(1)}}{\mu^{(1)}} + \frac{1}{\mu^{(2)}}, \quad m_2 := \frac{\tilde{\kappa}^{(2)}}{\mu^{(2)}} + \frac{1}{\mu^{(1)}}.$$

And now taking into account (2.3), it follows from (4.3) that

$$\lim_{x_2\to 0+}\Big\{\frac{1}{m_1}\Big(\Psi(z) + \frac{2}{\mu^{(2)}}\Phi(z)\Big) + \frac{1}{m_2}\Big(\Psi(\overline{z}) - \frac{2\tilde{\kappa}^{(2)}}{\mu^{(2)}}\Phi(\overline{z})\Big)\Big\} = 0 \quad \text{on } B_\varrho \cap \Gamma.$$

Since $\Phi(z)$ is continuous on $B_\varrho \cap \Gamma$, (4.4) yields

$$\Psi(z) = m_1\varphi^{(1)'}(z) - \frac{2}{\mu^{(2)}}\Phi(z) \quad \text{in } B_\varrho^{(1)},$$

and we obtain the Riemann-Hilbert problem

$$m_2\varphi^{(1)'}(z) + m_1\varphi^{(1)'}(\overline{z}) = 2\frac{\tilde{\kappa}^{(2)}+1}{\mu^{(2)}}\Phi(z) \quad \text{on } B_\varrho \cap \Gamma. \tag{4.8}$$

Then, the general solution for the homogeneous equation of (4.8) can be given by $\chi(z)X(z)$, where $\chi(z)$ is holomorphic on the whole B_ϱ and

$$X(z) := z^{-\gamma}(z+\varrho)^{\gamma-1}.$$

Note here that $X(z)$ is defined in the whole plane and has branch points at $z=0$, $z=-\varrho$. In order to define $X(z)$ uniquely we define $\arg z$ and $\arg(z+\varrho)$ as $-\pi < \arg z, \arg(z+\varrho) < \pi$. Then it is easy to see that $X(z)$ is holomorphic in the whole plane cut along $B_\varrho \cap \Gamma$. Now let $X^+(z) := \lim_{x_2\to 0+} X(z)$, $X^-(z) := \lim_{x_2\to 0-} X(z)$. We see that on $B_\varrho \cap \Gamma$

$$X^+(z) - \mathrm{e}^{-2\pi\mathrm{i}\gamma}X^-(z) = 0.$$

Consequently, we choose γ such that $m_1/m_2 = -\mathrm{e}^{-2\pi\mathrm{i}\gamma}$, that is,

$$-2\pi\mathrm{i}\gamma = \ln(m_1/m_2) + \mathrm{i}\arg(-m_1/m_2).$$

Since we seek a homogeneous solution of (4.8) in $L^2(B_\varrho)$, we take

$$\gamma := \frac{\mathrm{i}}{2\pi}\ln\Big(\frac{m_1}{m_2}\Big) + \frac{1}{2}.$$

Hence, $X(z)$ is a homogeneous solution of (4.8) and sectionally holomorphic in B_ϱ cut along $B_\varrho \cap \Gamma$ as required.

Furthermore, we can show that $\chi(z)X(z)$ is the general solution of the homogeneous equation of (4.8): $\varphi^{(1)'}(z) + (m_1/m_2)\varphi^{(1)'}(\overline{z}) = 0$. Since $X^+(z) + (m_1/m_2)X^-(z) = 0$, we have

$$\frac{\varphi^{(1)'}(z)}{X^+(z)} = \frac{\varphi^{(1)'}(\overline{z})}{X^-(z)} \quad \text{on } B_\varrho \cap \Gamma$$

and thus the function $\chi(z) := \varphi^{(1)'}(z)/X(z)$ is holomorphic in the whole B_ϱ. Similarly to the case of the inhomogeneous equation (4.8), we have

$$\frac{\varphi^{(1)'}(z)}{X^+(z)} - \frac{\varphi^{(1)'}(\overline{z})}{X^-(z)} = 2\frac{\tilde{\kappa}^{(2)}+1}{m_2\mu^{(2)}}\frac{\Phi(z)}{X^+(z)} \quad \text{on } B_\varrho \cap \Gamma.$$

Then, by virtue of the Plemelj formula (e.g., [9], [17], [27]) the general solution of (4.8) can be given by

$$\varphi^{(1)'}(z) = \frac{X(z)}{2\pi\mathrm{i}}\int_{B_\varrho\cap\Gamma} 2\frac{\tilde{\kappa}^{(2)}+1}{m_2\mu^{(2)}}\frac{\Phi(t)}{X^+(t)(t-z)}\,\mathrm{d}t + X(z)\chi(z). \tag{4.9}$$

The integral in (4.9) can be calculated by Cauchy's integral theorem and thus

$$\varphi^{(1)'}(z) = 2\frac{\tilde{\kappa}^{(2)}+1}{(m_1+m_2)\mu^{(2)}}\Phi(z) + X(z)\chi(z).$$

Indeed, it is obvious that $2(\tilde{\kappa}^{(2)}+1)/((m_1+m_2)\mu^{(2)})\Phi(z)$ is a special solution of (4.8). Resetting $\chi(z)$ defined in $B_{\varrho'}$ with $\varrho' < \varrho$ gives

$$\varphi^{(1)'}(z) = \mathrm{e}^{-\pi\varepsilon}z^{-1/2-\mathrm{i}\varepsilon}\chi(z) + 2\frac{\tilde{\kappa}^{(2)}+1}{(m_1+m_2)\mu^{(2)}}\Phi(z) \tag{4.10}$$

with

$$\varepsilon := \frac{1}{2\pi}\ln\Big(\frac{m_1}{m_2}\Big).$$

By employing the Dundurs parameter

$$\beta := \frac{\mu^{(2)}(\tilde{\kappa}^{(1)}-1)-\mu^{(1)}(\tilde{\kappa}^{(2)}-1)}{\mu^{(2)}(\tilde{\kappa}^{(1)}+1)+\mu^{(1)}(\tilde{\kappa}^{(2)}+1)} = \frac{m_1-m_2}{m_1+m_2},$$

see [6], [7], ε can be rewritten as $\varepsilon = -\frac{1}{2}\pi^{-1}\ln((1-\beta)/(1+\beta))$. We see that β varies from $-1/2$ to $1/2$ and vanishes for identical materials or special materials. Analogously, from (4.5)–(4.7) we find the expressions of the other functions:

$$\omega^{(1)'}(z) = \mathrm{e}^{\pi\varepsilon} z^{-\frac{1}{2}+\mathrm{i}\varepsilon}\overline{\chi(\overline{z})} - 2\frac{\tilde{\kappa}^{(2)}+1}{(m_1+m_2)\mu^{(2)}}\overline{\Phi(\overline{z})}, \tag{4.11}$$

$$\varphi^{(2)'}(z) = \mathrm{e}^{\pi\varepsilon} z^{-\frac{1}{2}-\mathrm{i}\varepsilon}\chi(z) + 2\frac{\tilde{\kappa}^{(1)}+1}{(m_1+m_2)\mu^{(1)}}\Phi(z), \tag{4.12}$$

$$\omega^{(2)'}(z) = \mathrm{e}^{-\pi\varepsilon} z^{-\frac{1}{2}+\mathrm{i}\varepsilon}\overline{\chi(\overline{z})} - 2\frac{\tilde{\kappa}^{(1)}+1}{(m_1+m_2)\mu^{(1)}}\overline{\Phi(\overline{z})}. \tag{4.13}$$

Lastly, we consider the non-penetration condition $[u_2] > 0$ on the crack.

From (4.1), $u_2^{(k)}$ on $B_\varrho \cap \Gamma$ can be represented as

$$4\mathrm{i}\mu^{(k)}u_2^{(k)} = \tilde{\kappa}^{(k)}(\varphi^{(k)}(z) - \overline{\varphi^{(k)}(z)}) - \overline{\omega^{(k)}(z)} + \omega^{(k)}(z). \tag{4.14}$$

Since $\chi(z)$ and $\Phi(z)$ are holomorphic on the whole $B_{\varrho'}$, they can be written as local Taylor series expansions

$$\chi(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \Phi(z) = \sum_{n=0}^{\infty} b_n z^n, \tag{4.15}$$

which are generalized uniformly convergent in $B_{\varrho'}$. Moreover, since the coefficients a_n, b_n can be given by

$$a_n = \frac{1}{2\pi\mathrm{i}}\int_{|w|=r}\frac{\chi(w)}{w^{n+1}}\,\mathrm{d}w \quad \text{and} \quad b_n = \frac{1}{2\pi\mathrm{i}}\int_{|w|=r}\frac{\Phi(w)}{w^{n+1}}\,\mathrm{d}w$$

for $0 < r < \varrho'$, by virtue of the Cauchy-Schwarz inequality it is easy to verify the estimates

$$|a_n| \leqslant c\sqrt{2n+1}\,(\varrho')^{-(n+\frac{1}{2})}\|\chi\|_{L^2(B_{\varrho'})},$$
$$|b_n| \leqslant c\sqrt{n+1}\,(\varrho')^{-(n+1)}\|\Phi\|_{L^2(B_{\varrho'})}.$$

By substituting (4.10)–(4.13) into (4.14) and using (4.15) the condition $[u_2] > 0$ on the crack can be reduced to a condition for the coefficient a_n, that is, on $B_{\varrho'} \cap \Gamma$,

$$\sum_{n=0}^{\infty}\frac{(-1)^n r^{\frac{1}{2}+n}}{(\frac{1}{2}+n)^2+\varepsilon^2}\Big\{\Big(\frac{1}{2}+n\Big)\operatorname{Re}[a_n r^{-\mathrm{i}\varepsilon}] - \varepsilon\operatorname{Im}[a_n r^{-\mathrm{i}\varepsilon}]\Big\} > 0. \tag{4.16}$$

Furthermore, let us set

$$\hat{a}_n := \frac{a_n}{\frac{1}{2}+n-\mathrm{i}\varepsilon}, \quad \hat{b}_n := \frac{b_n}{\mu^{(1)}\mu^{(2)}(m_1+m_2)(n+1)}.$$

Then (4.16) is rewritten as

$$\sum_{n=0}^{\infty}(-1)^n r^{\frac{1}{2}+n}\operatorname{Re}[\hat{a}_n r^{-\mathrm{i}\varepsilon}] > 0. \tag{4.17}$$

Summing up the above gives the convergent expansion of $\boldsymbol{u}^{(k)}$ near the crack tip.

Proposition 4.1. *For Case* 1 *there exist complex numbers $\hat{a}_n$ satisfying the condition* (4.17)*, $\hat{b}_n$, and a constant vector $\boldsymbol{c}$ such that for $k=1,2$*

$$\begin{aligned}\boldsymbol{u}^{(k)}(r,\theta) = &\sum_{n=0}^{\infty}\frac{\mathrm{e}^{(-1)^k\varepsilon\pi}r^{\frac{1}{2}+n}}{2\mu^{(k)}}\big\{\operatorname{Re}[\hat{a}_n r^{-\mathrm{i}\varepsilon}]\boldsymbol{P}^{(k)}_{1,n}(\theta) - \operatorname{Im}[\hat{a}_n r^{-\mathrm{i}\varepsilon}]\boldsymbol{Q}^{(k)}_{1,n}(\theta)\big\}\\ &+\sum_{n=0}^{\infty} r^{n+1}\tilde{d}_k\big\{\operatorname{Re}[\hat{b}_n]\boldsymbol{R}^{(k)}_{1,n}(\theta) - \operatorname{Im}[\hat{b}_n]\boldsymbol{S}^{(k)}_{1,n}(\theta)\big\} + \boldsymbol{F}(\boldsymbol{x})\boldsymbol{c},\end{aligned}$$

where $\tilde{d}_1 = \tilde{\kappa}^{(2)}+1$, $\tilde{d}_2 = \tilde{\kappa}^{(1)}+1$,

$$\boldsymbol{P}^{(k)}_{1,n}(\theta) = \mathrm{e}^{\varepsilon\theta}\begin{pmatrix} (\tilde{\kappa}^{(k)}+n+\frac{1}{2}-\mathrm{e}^{-2\varepsilon(\theta+(-1)^k\pi)})\cos(n+\frac{1}{2})\theta \\ +\varepsilon(\sin(n+\frac{1}{2})\theta - \sin(n-\frac{3}{2})\theta) - (n+\frac{1}{2})\cos(n-\frac{3}{2})\theta \\ (\tilde{\kappa}^{(k)}-n-\frac{1}{2}+\mathrm{e}^{-2\varepsilon(\theta+(-1)^k\pi)})\sin(n+\frac{1}{2})\theta \\ +\varepsilon(\cos(n+\frac{1}{2})\theta - \cos(n-\frac{3}{2})\theta) + (n+\frac{1}{2})\sin(n-\frac{3}{2})\theta \end{pmatrix},$$

$$\boldsymbol{Q}^{(k)}_{1,n}(\theta) = \mathrm{e}^{\varepsilon\theta}\begin{pmatrix} (\tilde{\kappa}^{(k)}+n+\frac{1}{2}+\mathrm{e}^{-2\varepsilon(\theta+(-1)^k\pi)})\sin(n+\frac{1}{2})\theta \\ -\varepsilon(\cos(n+\frac{1}{2})\theta - \cos(n-\frac{3}{2})\theta) - (n+\frac{1}{2})\sin((n-\frac{3}{2})\theta) \\ (-\tilde{\kappa}^{(k)}+n+\frac{1}{2}+\mathrm{e}^{-2\varepsilon(\theta+(-1)^k\pi)})\cos(n+\frac{1}{2})\theta \\ +\varepsilon(\sin(n+\frac{1}{2})\theta - \sin(n-\frac{3}{2})\theta) - (n+\frac{1}{2})\cos(n-\frac{3}{2})\theta \end{pmatrix},$$

$$\boldsymbol{R}^{(k)}_{1,n}(\theta) = \begin{pmatrix} \tilde{\kappa}^{(k)}\cos(n+1)\theta - (n+1)\cos(n-1)\theta + (n+2)\cos(n+1)\theta \\ \tilde{\kappa}^{(k)}\sin(n+1)\theta + (n+1)\sin(n-1)\theta - (n+2)\sin(n+1)\theta \end{pmatrix},$$

$$\boldsymbol{S}^{(k)}_{1,n}(\theta) = \begin{pmatrix} \tilde{\kappa}^{(k)}\sin(n+1)\theta - (n+1)\sin(n-1)\theta + n\sin(n+1)\theta \\ -\tilde{\kappa}^{(k)}\cos(n+1)\theta - (n+1)\cos(n-1)\theta + n\cos(n+1)\theta \end{pmatrix}.$$

The series are convergent, absolutely in $H^1(B^{(k)}_{\varrho'})$ and generalized uniformly in $B^{(k)}_{\varrho'}$ for $k=1,2$, respectively. For $n \geqslant 0$, $\hat{a}_n$ and $\hat{b}_n$ satisfy

$$|\hat{a}_n| \leqslant c\frac{1}{\sqrt{2n+1}}\,(\varrho')^{-(n+\frac{1}{2})}\|\nabla\boldsymbol{u}\|_{L^2(B_{\varrho'})},$$
$$|\hat{b}_n| \leqslant c\frac{1}{\sqrt{n+1}}\,(\varrho')^{-(n+1)}\|\nabla\boldsymbol{u}\|_{L^2(B_{\varrho'})}.$$

Note that the estimates of coefficients can be obtained from (4.2)–(4.3), and the coefficients of the leading terms in the expansion are called, in fracture mechanics, the stress intensity factors. In the case of homogeneous material which means $\varepsilon = 0$ the formula in Proposition 4.1 coincides with the form in [15], [26] and, furthermore, the condition (4.17) implies nonnegativity of $\mathrm{Re}[\hat{a}_0]$, which corresponds to the results in [3], [22].

4.2. Case 2(a) (stick state)

In this case, first, it follows from the problem $(*)$ and (2.4) that

$$[\sigma_{22} - \mathrm{i}\sigma_{12}] = 0 \quad \text{and} \quad [\boldsymbol{u}] = \mathbf{0} \quad \text{on } B_\varrho \cap \Gamma'.$$

Consequently, in a way exactly similar to Case 1 we can construct functions $\varphi^{(k)}(z)$, $\omega^{(k)}(z)$ $(k = 1, 2)$ satisfying (4.4)–(4.7). However, in contrast to Case 1, $\Psi(z)$ is continuous on $B_\varrho \cap \Gamma'$. Namely, both $\Phi(z)$ and $\Psi(z)$ are holomorphic in B_ϱ. Hence, they can be written as local Taylor series expansions

$$\Psi(z) = \sum_{n=0}^{\infty} c_n z^n, \quad \Phi(z) = \sum_{n=0}^{\infty} b_n z^n, \tag{4.18}$$

which are generalized uniformly convergent in B_ϱ.

Second, since it follows from (4.3) that on $B_\varrho \cap \Gamma'$

$$\sigma_{22}^{(k)} = \frac{1}{2}\{\varphi^{(k)\prime}(z) + \overline{\varphi^{(k)\prime}(z)} + \overline{\omega^{(k)\prime}(z)} + \omega^{(k)\prime}(z)\} \tag{4.19}$$

and

$$\sigma_{12}^{(k)} = \frac{\mathrm{i}}{2}\{\varphi^{(k)\prime}(z) - \overline{\varphi^{(k)\prime}(z)} + \overline{\omega^{(k)\prime}(z)} - \omega^{(k)\prime}(z)\}, \tag{4.20}$$

one can see that the condition (2.5) is equivalent to the condition on $B_\varrho \cap \Gamma$

$$\sum_{n=0}^{\infty} r^n (-1)^n \Big\{(m_1 + m_2)\,\mathrm{Re}[c_n] - \frac{2(\tilde{\kappa}^{(1)}\tilde{\kappa}^{(2)} - 1)}{\mu^{(1)}\mu^{(2)}}\,\mathrm{Re}[b_n]\Big\} \leqslant 0. \tag{4.21}$$

Moreover, since (4.21) is valid as r tends to 0, one has

$$(m_1 + m_2)\,\mathrm{Re}[c_0] - \frac{2(\tilde{\kappa}^{(1)}\tilde{\kappa}^{(2)} - 1)}{\mu^{(1)}\mu^{(2)}}\,\mathrm{Re}[b_0] \leqslant 0. \tag{4.22}$$

On the other hand, one knows that the condition (2.6) is reduced to a condition on $B_\varrho \cap \Gamma$,

$$\Big|\sum_{n=0}^{\infty} r^n(-1)^{n+1}\Big\{(m_1+m_2)\,\mathrm{Im}[c_n] - \frac{2(\tilde\kappa^{(1)}\tilde\kappa^{(2)}-1)}{\mu^{(1)}\mu^{(2)}}\,\mathrm{Im}[b_n]\Big\}\Big| \leqslant -f\sum_{n=0}^{\infty} r^n(-1)^{n}\Big\{(m_1+m_2)\,\mathrm{Re}[c_n] - \frac{2(\tilde\kappa^{(1)}\tilde\kappa^{(2)}-1)}{\mu^{(1)}\mu^{(2)}}\,\mathrm{Re}[b_n]\Big\} \tag{4.23}$$

and also we have

$$\Big|(m_1+m_2)\,\mathrm{Im}[c_0] - \frac{2(\tilde\kappa^{(1)}\tilde\kappa^{(2)}-1)}{\mu^{(1)}\mu^{(2)}}\,\mathrm{Im}[b_0]\Big| \leqslant -f\Big\{(m_1+m_2)\,\mathrm{Re}[c_0] - \frac{2(\tilde\kappa^{(1)}\tilde\kappa^{(2)}-1)}{\mu^{(1)}\mu^{(2)}}\,\mathrm{Re}[b_0]\Big\}. \tag{4.24}$$

Next, by substituting (4.4)–(4.7) into (4.1) and using (4.18) we obtain the convergent expansion of $\boldsymbol{u}^{(k)}$ near the crack tip.

Proposition 4.2. *For Case* 2(a) *there exist complex numbers* c_n, b_n *satisfying the conditions* (4.21)–(4.24) *and a constant vector* $\boldsymbol{c}$ *such that for* $k=1,2$

$$\boldsymbol{u}^{(k)}(r,\theta) = \sum_{n=0}^{\infty}\frac{r^{n+1}}{2\mu^{(k)}m_k(n+1)}\{\mathrm{Re}[c_n]\boldsymbol{P}^{(k)}_{2a,n}(\theta) - \mathrm{Im}[c_n]\boldsymbol{Q}^{(k)}_{2a,n}(\theta)\} + \sum_{n=0}^{\infty}\frac{r^{n+1}}{\mu^{(1)}\mu^{(2)}(n+1)}\{\mathrm{Re}[b_n]\boldsymbol{R}^{(k)}_{2a,n}(\theta) - \mathrm{Im}[b_n]\boldsymbol{S}^{(k)}_{2a,n}(\theta)\} + \boldsymbol{F}(\boldsymbol{x})\boldsymbol{c},$$

where

$$\boldsymbol{P}^{(k)}_{2a,n}(\theta) = \begin{pmatrix} (\tilde\kappa^{(k)}+n+1-\mathrm{e}^{(-1)^{k+1}2\pi\varepsilon})\cos(n+1)\theta - (n+1)\cos(n-1)\theta \\ (\tilde\kappa^{(k)}-n-1+\mathrm{e}^{(-1)^{k+1}2\pi\varepsilon})\sin(n+1)\theta + (n+1)\sin(n-1)\theta \end{pmatrix},$$

$$\boldsymbol{Q}^{(k)}_{2a,n}(\theta) = \begin{pmatrix} (\tilde\kappa^{(k)}+n+1+\mathrm{e}^{(-1)^{k+1}2\pi\varepsilon})\sin(n+1)\theta - (n+1)\sin(n-1)\theta \\ (-\tilde\kappa^{(k)}+n+1+\mathrm{e}^{(-1)^{k+1}2\pi\varepsilon})\cos(n+1)\theta - (n+1)\cos(n-1)\theta \end{pmatrix},$$

$$\boldsymbol{R}^{(k)}_{2a,n}(\theta) = \begin{pmatrix} \big(\frac{\tilde\kappa^{(1)}}{m_1}+\frac{\tilde\kappa^{(2)}}{m_2}+\frac{n+1}{m_k}\big)\cos(n+1)\theta - \frac{n+1}{m_k}\cos(n-1)\theta \\ \big((-1)^{k+1}\frac{\tilde\kappa^{(1)}}{m_1}+(-1)^{k}\frac{\tilde\kappa^{(2)}}{m_2}-\frac{n+1}{m_k}\big)\sin(n+1)\theta + \frac{n+1}{m_k}\sin(n-1)\theta \end{pmatrix},$$

$$\boldsymbol{S}^{(k)}_{2a,n}(\theta) = \begin{pmatrix} \big((-1)^{k+1}\frac{\tilde\kappa^{(1)}}{m_1}+(-1)^{k}\frac{\tilde\kappa^{(2)}}{m_2}+\frac{n+1}{m_k}\big)\sin(n+1)\theta - \frac{n+1}{m_k}\sin(n-1)\theta \\ \big(-\frac{\tilde\kappa^{(1)}}{m_1}-\frac{\tilde\kappa^{(2)}}{m_2}+\frac{n+1}{m_k}\big)\cos(n+1)\theta - \frac{n+1}{m_k}\cos(n-1)\theta \end{pmatrix}.$$

The series are convergent, absolutely in $H^1(B^{(k)}_\varrho)$ *and generalized uniformly in* $B^{(k)}_\varrho$ *for* $k=1,2$*, respectively. For* $n\geqslant 0$, c_n *and* b_n *satisfy*

$$|c_n| \leqslant c\sqrt{n+1}\,(\varrho')^{-(n+1)}\|\nabla\boldsymbol{u}\|_{L^2(B_{\varrho'})},$$
$$|b_n| \leqslant c\sqrt{n+1}\,(\varrho')^{-(n+1)}\|\nabla\boldsymbol{u}\|_{L^2(B_{\varrho'})}.$$

4.3. Case 2(b) (slip state)

In this case, first, it follows from the problem ($*$) and (2.7) that

$$[\sigma_{22} - \mathrm{i}\sigma_{12}] = 0 \quad \text{on } B_\varrho \cap \Gamma', \qquad [\boldsymbol{u}] = \boldsymbol{0} \quad \text{on } B_\varrho \cap (\Gamma' \setminus \overline{\Gamma}).$$

Consequently, in a way exactly similar to Case 1 we can construct functions $\varphi^{(k)}(z)$, $\omega^{(k)}(z)$ $(k = 1, 2)$ satisfying (4.4)–(4.7).

Second, taking into account the condition $[u_2] = 0$ on $B_\varrho \cap \Gamma$, it follows from (4.1) that on $B_\varrho \cap \Gamma$

$$\frac{\tilde{\kappa}^{(1)}}{\mu^{(1)}}\big(\varphi^{(1)}(x_1) - \overline{\varphi^{(1)}(x_1)}\big) - \frac{1}{\mu^{(1)}}\big(\overline{\omega^{(1)}(x_1)} - \omega^{(1)}(x_1)\big)$$
$$- \frac{\tilde{\kappa}^{(2)}}{\mu^{(2)}}\big(\varphi^{(2)}(x_1) - \overline{\varphi^{(2)}(x_1)}\big) + \frac{1}{\mu^{(2)}}\big(\overline{\omega^{(2)}(x_1)} - \omega^{(2)}(x_1)\big) = 0.$$

Differentiating both sides of this equality with respect to x_1 yields

$$\frac{\tilde{\kappa}^{(1)}}{\mu^{(1)}}\big(\varphi^{(1)\prime}(x_1) - \overline{\varphi^{(1)\prime}(x_1)}\big) - \frac{1}{\mu^{(1)}}\big(\overline{\omega^{(1)\prime}(x_1)} - \omega^{(1)\prime}(x_1)\big)$$
$$- \frac{\tilde{\kappa}^{(2)}}{\mu^{(2)}}\big(\varphi^{(2)\prime}(x_1) - \overline{\varphi^{(2)}(x_1')}\big) + \frac{1}{\mu^{(2)}}\big(\overline{\omega^{(2)\prime}(x_1)} - \omega^{(2)\prime}(x_1)\big) = 0.$$

This implies that

$$\lim_{x_2 \to 0+} \Psi(z) + \overline{\Psi(\overline{z})} = \lim_{x_2 \to 0-} \Psi(z) + \overline{\Psi(\overline{z})} \quad \text{on } B_\varrho \cap \Gamma.$$

Thus the function $\Psi(z) + \overline{\Psi(\overline{z})}$ is holomorphic on the whole B_ϱ, therefore it is defined by a holomorphic function $\check{\Psi}(z)$,

$$\check{\Psi}(z) := \frac{1}{2}\big(\Psi(z) + \overline{\Psi(\overline{z})}\big). \tag{4.25}$$

Next, taking into account (2.9), by using (4.19) and (4.20) we have

$$\lim_{x_2 \to 0+} \big\{\mathrm{i}\big(\varphi^{(1)\prime}(z) - \overline{\varphi^{(1)\prime}(z)} + \overline{\omega^{(1)\prime}(z)} - \omega^{(1)\prime}(z)\big) \pm f\big(\varphi^{(1)\prime}(z) + \overline{\varphi^{(1)\prime}(z)} + \overline{\omega^{(1)\prime}(z)} + \omega^{(1)\prime}(z)\big)\big\} = 0 \tag{4.26}$$

on $B_\varrho \cap \Gamma$, where the upper sign is taken for the case $[u_1] > 0$ and the lower sign is taken for the case $[u_1] < 0$. Then it follows from (4.4)–(4.7) that (4.26) can be represented as

$$\lim_{x_2 \to 0+} \Big\{\frac{\mathrm{i} \pm f}{m_1}\Psi(z) + \frac{-\mathrm{i} \pm f}{m_1}\overline{\Psi(z)} + \frac{\mathrm{i} \pm f}{m_2}\Psi(\overline{z}) + \frac{-\mathrm{i} \pm f}{m_2}\overline{\Psi(\overline{z})} + \frac{2}{\mu^{(2)}}\Big(\frac{1}{m_1} - \frac{\tilde{\kappa}^{(2)}}{m_2}\Big)\big((\mathrm{i} \pm f)\Phi(z) + (-\mathrm{i} \pm f)\overline{\Phi(\overline{z})}\big)\Big\} = 0 \tag{4.27}$$

on $B_\varrho \cap \Gamma$. Since it follows from (4.25) that (4.27) leads to

$$\begin{aligned}(m_2(\mathrm{i} \pm f) - m_1(-\mathrm{i} \pm f))\Psi(z) &+ (m_1(\mathrm{i} \pm f) - m_2(-\mathrm{i} \pm f))\Psi(\overline{z}) \\ = &- 2(-\mathrm{i} \pm f)(m_1 + m_2)\check{\Psi}(z) \\ &+ \frac{2}{\mu^{(2)}}(m_1\tilde{\kappa}^{(2)} - m_2)\big((\mathrm{i} \pm f)\Phi(z) + (-\mathrm{i} \pm f)\overline{\Phi(\overline{z})}\big)\end{aligned}$$

on $B_\varrho \cap \Gamma$, from (4.4) we obtain the Riemann-Hilbert problem

$$\varphi^{(1)'}(z) + \check{f}\varphi^{(1)'}(\overline{z}) = \frac{\check{\Phi}_2(z)}{m_1 + m_2 \pm \mathrm{i}f(m_1 - m_2)} \quad \text{on } B_\varrho \cap \Gamma, \tag{4.28}$$

where

$$\check{f} := \frac{m_1 + m_2 \mp \mathrm{i}f(m_1 - m_2)}{m_1 + m_2 \pm \mathrm{i}f(m_1 - m_2)} = \frac{1 \mp \mathrm{i}f\beta}{1 \pm \mathrm{i}f\beta}$$

and $\check{\Phi}_k(z)$ $(k = 1, 2)$ is a holomorphic function in B_ϱ defined as

$$\begin{aligned}\check{\Phi}_k(z) := 2(1 \pm \mathrm{i}f)(1 + \mathrm{e}^{(-1)^{k-1}2\pi\varepsilon})\check{\Psi}(z) &+ \frac{2}{\mu^{(k)}}(\tilde{\kappa}^{(k)} + 2 + \mathrm{e}^{(-1)^{k-1}2\pi\varepsilon})\Phi(z) \\ - \frac{2}{\mu^{(k)}}(\tilde{\kappa}^{(k)} - \mathrm{e}^{(-1)^{k-1}2\pi\varepsilon})&\big((1 \pm \mathrm{i}f)\overline{\Phi(\overline{z})} \pm \mathrm{i}f\Phi(z)\big).\end{aligned}$$

In a way similar to solving (4.8), we obtain the general solution of (4.28) given by

$$\varphi^{(1)'}(z) = \check{X}(z)\chi(z) + \frac{\check{\Phi}_2(z)}{2(m_1 + m_2)}, \tag{4.29}$$

where $\chi(z)$ is a holomorphic function in B_ϱ and

$$\check{X}(z) := z^{-\check{\gamma}}(z + \varrho)^{\check{\gamma}-1}$$

with

$$\check{\gamma} := \frac{\mathrm{i}}{2\pi}\ln|\check{f}| - \frac{1}{2\pi}\arg(-\check{f}).$$

Since it is easy to see that $|\check{f}| = 1$ and thus

$$\cos(-2\pi\check{\gamma}) = \mathrm{Re}[-\check{f}] = -\frac{1 - f^2\beta^2}{1 + f^2\beta^2}$$

and $\sin(-2\pi\check{\gamma}) = \mathrm{Im}[-\check{f}] = \pm 2f\beta/(1 + f^2\beta^2)$, we have

$$\cot(\pi\check{\gamma}) = \frac{1 + \cos(2\pi\check{\gamma})}{\sin(2\pi\check{\gamma})} = \mp f\beta, \tag{4.30}$$

where the upper sign "$-$" is for the case $[u_1] > 0$ on Γ and the lower sign "$+$" is for the case $[u_1] < 0$ on Γ. Therefore, since the given f is assumed to be less than 1 (see (3.11)), β varies from $-1/2$ to $1/2$ and $\varphi^{(k)'}(z) \in L^2(B_\varrho^{(k)})$, we can uniquely choose $\check{\gamma} \in \mathbb{R}$ satisfying (4.30) and $0 < \check{\gamma} < 1$. In fact, a possibility of the case $\frac{1}{2} < \check{\gamma} < 1$ is precluded by inequality conditions on Γ, for the details see the end of this section. And according to [2], it is shown that $\check{\gamma}$ cannot be larger than $\frac{1}{2}$ by excluding an inconsistent situation of a backward propagation of the crack, see also [7]. Moreover, note that $\check{\gamma} = \frac{1}{2}$ if and only if $\beta = 0$, which includes identical materials.

Next, by resetting $\chi(z)$ in $B_{\varrho'}$ for $\varrho' < \varrho$, (4.29) is rewritten as

$$\varphi^{(1)'}(z) = \mathrm{e}^{-\pi\varepsilon} z^{-\check{\gamma}} \chi(z) + \frac{\check{\Phi}_2(z)}{2(m_1 + m_2)}. \tag{4.31}$$

Now (4.5)–(4.7) yield

$$\omega^{(1)'}(z) = \mathrm{e}^{\pi\varepsilon} z^{-\check{\gamma}} \overline{\chi(\overline{z})} + \frac{\overline{\check{\Phi}_1(\overline{z})}}{2(m_1 + m_2)} - 2\overline{\Phi(\overline{z})}, \tag{4.32}$$

$$\varphi^{(2)'}(z) = \mathrm{e}^{\pi\varepsilon} z^{-\check{\gamma}} \chi(z) + \frac{\check{\Phi}_1(z)}{2(m_1 + m_2)}, \tag{4.33}$$

$$\omega^{(2)'}(z) = \mathrm{e}^{-\pi\varepsilon} z^{-\check{\gamma}} \overline{\chi(\overline{z})} + \frac{\overline{\check{\Phi}_2(\overline{z})}}{2(m_1 + m_2)} - 2\overline{\Phi(\overline{z})}. \tag{4.34}$$

Since $\Phi(z)$, $\chi(z)$, and $\check{\Psi}(z)$ are holomorphic in $B_{\varrho'}$ they can be written as local Taylor series expansions

$$\Phi(z) = \sum_{n=0}^{\infty} b_n z^n, \quad \chi(z) = \sum_{n=0}^{\infty} \check{a}_n z^n, \quad \check{\Psi}(z) = \sum_{n=0}^{\infty} \check{c}_n z^n, \tag{4.35}$$

which are generalized uniformly convergent in $B_{\varrho'}$. However, $\check{\Psi}(z)$ must be holomorphic on the whole $B_{\varrho'}$, and by using (4.31) one can see that $\mathrm{Re}[\check{a}_n] = \mathrm{Im}[\check{c}_n] = 0$ for every $n \geqslant 0$.

Then, from (4.19) one can see that the condition (2.8) is equivalent to the following condition on $B_{\varrho'} \cap \Gamma$:

$$\sum_{n=0}^{\infty} (-1)^n \Big\{ r^{n-\check{\gamma}} (\mathrm{e}^{-\varepsilon\pi} - \mathrm{e}^{\varepsilon\pi}) \sin \check{\gamma}\pi \, \mathrm{Im}[\check{a}_n] - r^n \frac{2(\tilde{\kappa}^{(1)}\tilde{\kappa}^{(2)} - 1)}{\mu^{(1)}\mu^{(2)} m_1 m_2} \mathrm{Re}[b_n] \Big\} + \sum_{n=0}^{\infty} (-1)^n r^n \frac{1}{2} \Big(\frac{1}{m_1} + \frac{1}{m_2} \Big) \mathrm{Re}[\check{c}_n] \leqslant 0. \tag{4.36}$$

It follows from this and $\check{\gamma} > 0$ that

$$(\mathrm{e}^{-\varepsilon\pi} - \mathrm{e}^{\varepsilon\pi})\,\mathrm{Im}[\check{a}_0] = \frac{-2\beta}{\sqrt{1-\beta^2}}\,\mathrm{Im}[\check{a}_0] \leqslant 0. \tag{4.37}$$

Furthermore, by substituting (4.31)–(4.34) into (4.1) and using (4.35) we obtain

$$[u_1] = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} r^{n+1-\check{\gamma}}}{n+1-\check{\gamma}} \sqrt{m_1 m_2}\,\sin\check{\gamma}\pi\,\mathrm{Im}[\check{a}_n] \quad \text{on } B_{\varrho'} \cap \Gamma. \tag{4.38}$$

Summing up the above, we have the convergent expansion of $\boldsymbol{u}^{(k)}$ near the crack tip as follows.

Proposition 4.3. *For Case* 2(b) *there exist complex numbers* $\check{a}_n$, b_n, $\check{c}_n$ *satisfying the condition* (4.36), *and a constant vector* $\boldsymbol{c}$ *such that for* $k = 1, 2$

$$\begin{aligned}
\boldsymbol{u}^{(k)}(r,\theta) = {} & \sum_{n=0}^{\infty} \frac{\mathrm{e}^{(-1)^k \varepsilon\pi} r^{n+1-\check{\gamma}}}{2\mu^{(k)}(n+1-\check{\gamma})} \{-\mathrm{Im}[\check{a}_n] \boldsymbol{Q}^{(k)}_{2b,n}(\theta)\} \\
& + \sum_{n=0}^{\infty} \frac{r^{n+1}}{\mu^{(1)}\mu^{(2)}(n+1)} \mathrm{Re}[b_n] \Big(\boldsymbol{R}^{(k)}_{2a,n}(\theta) \pm \frac{f(\tilde{\kappa}^{(1)}\tilde{\kappa}^{(2)} - 1)}{\mu^{(k)} m_k (m_1+m_2)} \boldsymbol{Q}^{(k)}_{2a,n}(\theta) \Big) \\
& - \sum_{n=0}^{\infty} \frac{r^{n+1} \tilde{d}_k}{\mu^{(1)}\mu^{(2)}(m_1+m_2)(n+1)} \mathrm{Im}[b_n] \boldsymbol{S}^{(k)}_{1,n}(\theta) \\
& + \sum_{n=0}^{\infty} \frac{r^{n+1}}{2\mu^{(k)} m_k (n+1)} \mathrm{Re}[\check{c}_n] \{\boldsymbol{P}^{(k)}_{2a,n}(\theta) \mp f \boldsymbol{Q}^{(k)}_{2a,n}(\theta)\} + \boldsymbol{F}(\boldsymbol{x})\boldsymbol{c},
\end{aligned}$$

where the upper and lower signs are taken when (4.38) *is positive and negative, respectively,*

$$\boldsymbol{Q}^{(k)}_{2b,n}(\theta) = \begin{pmatrix} (\tilde{\kappa}^{(k)} + n + 1 - \check{\gamma} + \mathrm{e}^{(-1)^{k+1} 2\pi\varepsilon}) \sin(n+1-\check{\gamma})\theta - (n+1-\check{\gamma}) \sin(n-1-\check{\gamma})\theta \\ (-\tilde{\kappa}^{(k)} + n + 1 - \check{\gamma} + \mathrm{e}^{(-1)^{k+1} 2\pi\varepsilon}) \cos(n+1-\check{\gamma})\theta - (n+1-\check{\gamma}) \cos(n-1-\check{\gamma})\theta \end{pmatrix}.$$

The series are convergent, absolutely in $H^1(B^{(k)}_{\varrho'})$ *and generalized uniformly in* $B^{(k)}_{\varrho'}$ *for* $k = 1, 2$, *respectively. For* $n \geqslant 0$, $\check{a}_n$, b_n, *and* $\check{c}_n$ *satisfy*

$$\begin{aligned}
|\check{a}_n| &\leqslant c\sqrt{n+1-\check{\gamma}}\,(\varrho')^{-(n+1-\check{\gamma})} \|\nabla \boldsymbol{u}\|_{L^2(B_{\varrho'})}, \\
|b_n| &\leqslant c\sqrt{n+1}\,(\varrho')^{-(n+1)} \|\nabla \boldsymbol{u}\|_{L^2(B_{\varrho'})}, \\
|\check{c}_n| &\leqslant c\sqrt{n+1}\,(\varrho')^{-(n+1)} \|\nabla \boldsymbol{u}\|_{L^2(B_{\varrho'})}.
\end{aligned}$$

Note that $\check{\gamma}$ can vary from 0 to 1, which implies a possibility of a stronger singularity of the stress at the crack tip than the inverse square root. However, this case can be precluded by the following reason. Let us assume $0 < \check{\gamma} < 1$ and $\beta \neq 0$. If $[u_1] > 0$ on $B_\varrho \cap \Gamma$, then $\cot \check{\gamma}\pi = -f\beta$ and $\operatorname{Im}[\check{a}_0] \leqslant 0$ by (4.38). Combining this with (4.37), one sees that $\beta < 0$ or $\operatorname{Im}[\check{a}_0] = 0$, which means the singular term of the expansion disappears. Then $\beta < 0$ implies $-f\beta > 0$ and thus we conclude $0 < \check{\gamma} < \frac{1}{2}$.

Similarly, if $[u_1] < 0$ on $B_\varrho \cap \Gamma$, then $\cot \check{\gamma}\pi = f\beta$ and $\operatorname{Im}[\check{a}_0] \geqslant 0$ by (4.38). Combining this with (4.37), one sees that $\beta > 0$ or $\operatorname{Im}[\check{a}_0] = 0$ and thus we conclude $0 < \check{\gamma} < \frac{1}{2}$. In the case $\beta = 0$, one knows $\check{\gamma} = \frac{1}{2}$ by (4.30) and from (4.38) we have $\operatorname{Im}[\check{a}_0] \leqslant 0$ for $[u_1] > 0$ on $B_\varrho \cap \Gamma$ and $\operatorname{Im}[\check{a}_0] \geqslant 0$ for $[u_1] < 0$ on $B_\varrho \cap \Gamma$.

5. Conclusion

We derived the complete asymptotic expansions of the displacement near the tip of the crack on the interface between two dissimilar elastic media, written in Proposition 4.1–4.3 under each one of the following three conditions: open crack, stick state, slip state. It assumes the exact forms with respect to the distance to the crack tip as well as the explicit expression of the angular functions around the crack tip. Under the assumption that there are no switches among the three possible cases, the expansion with the convergence proof is obtained in each case. Thus, it enables us to have an a priori regularity of the solution near the crack tip. Indeed, the open crack in Case 1 implies $\boldsymbol{u} \notin H^{3/2}(B_\varrho \setminus \overline{\Gamma})$, the solution is smooth in the stick state of Case 2(a), and for general dissimilar materials, i.e. $\beta \neq 0$, $\boldsymbol{u} \in H^{3/2}(B_\varrho \setminus \overline{\Gamma})$ in the slip state of Case 2(b). We also derive explicit conditions with respect to coefficients in the expansions arising from inequality type conditions on the crack, that is, non-penetration conditions which make our problem meaningful in the physical sense.

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