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Generalization of the Concept of the Topological Derivative for a Kinking Crack

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Abstract:

Arising in fracture mechanics and engineering, a model scalar-valued problem of kinking of a crack is investigated within the topological context. The objective function representing the potential energy is expanded with respect to the infinitesimal hole (micro-crack) at the tip of the macro-crack. Based on the refined method of matched asymptotic expansions, the asymptotic models are derived in the terms of stress intensity factors. In particular, this gives the so-called topological derivatives of the first-order, which were out of the case of the smooth conception of topological derivatives.

Keywords: fracture mechanics, kinking crack, topological derivative

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1. INTRODUCTION

Appearing in a wide range of real world applications, macro- and micro-cracking phenomena are of the primary interest for fracture mechanics and the related engineering problems in geo-, bio-, and material sciences. For the common fracture conceptions we refer to *(4, 8, 11)*, and other works. In particular, the problem of crack kinking including determining the direction in which an incipient crack will propagate is the subject for permanent discussions in the literature on fracture mechanics, see *(1, 2, 5, 9)*. In its almost a hundred year history fracture theory has addressed a great number of problems using a variety of techniques, but still there are many formal inconsistencies. In fact, the process of translating micro-effects onto the macro-level results in a singular character of mathematical models describing cracks. Such singular problems are difficult to analyse by numerical methods. Our study aims to investigate cracking phenomena within topological context.

In fact, a kink of the crack path during its evolution can be viewed as a stoppage of movement in the direction tangential to the previous smooth path and an appearance of a new path. This peculiarity of the crack propagation process is inherently connected with the phenomenon of appearance of a hole or a micro-crack in a continuum. From a geometric viewpoint, these phenomena present change of the topology class of the continuum. Describing changes in topology is the key difficulty in the structure analysis and optimization. Recently, a mathematical foundation

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of the topological sensitivity due to creating infinitesimal holes in a continuum was given in the form of the so-called bubble-method, see *(6, 13)*. The mathematical formalism exploits the corresponding topological derivative of the objective function (the potential energy in our context) when the hole vanishes. However, the main disadvantage of the standard approach concerns the fact that the topological derivative was defined for a-priori smooth geometries only. Pre-described non-smooth cracks inside the continuum are not the case. By these reasons, we propose a generalization of the topological derivative concept specifically for the non-smooth case represented by cracks.

To emphasize the main difficulties arising in the problem under consideration, we rely on a model scalar-valued problem and on a piecewise-linear crack path. Our approach is based on the refined method of matched asymptotic expansions for topological sensitivity analysis, see *(3,12)*. We derive asymptotic models for the objective function representing the potential energy with respect to the infinitesimal hole (micro-crack) at the tip of the macro-crack. In particular, this approach gives the first-order topological derivatives, which were out of the case of the smooth conception (which assumes that the first-order terms are zero for smooth problems).

2. ASYMPTOTIC ANALYSIS OF THE ENERGY FUNCTIONAL

2.1 Statement of the problem

Let $\Xi_\infty = \{x = (x_1, x_2) : x_1 \le 0, x_2 = 0\}$ be a semi-infinite crack on R^2 with two faces $\Xi_\infty^\pm$. Suppose that Γ is a simple contour with its ends lying on the faces $\Xi_\infty^\pm$ of the crack. Let Ω be a domain with the boundary formed by the contour Γ and the faces $\Xi_0^\pm$ of a finite crack Ξ_0 with its tip at the origin of coordinates O. We consider a hole ω containing the origin O with the piecewise-smooth boundary $\partial\omega$. In our special case when treating kinking cracks, we specify the hole ω as a linear segment (O,Q), and the boundary $\partial\omega$ turns into its two opposite faces with a normal vector defined well at the each face except at the tips O and Q. By ε we denote a small positive parameter. For sufficiently small ε it is always possible to remove the set $\omega_\varepsilon = \{x : \varepsilon^{-1}x \in \omega\}$ from the domain Ω, thus obtaining the singularly perturbed domain $\Omega_\varepsilon = \Omega \setminus \overline{\omega_\varepsilon}$ with the boundary $\partial\Omega_\varepsilon = \Gamma \bigcup \Xi_\varepsilon^\pm \bigcup \omega_\varepsilon$. In such a domain we consider the following Neumann boundary value problem for the Poisson equation:

$$-\Delta_x u^\varepsilon(x) = 0, \qquad x \in \Omega_\varepsilon; \qquad (1)$$

$$\partial u^\varepsilon / \partial n(x) = 0, \qquad x \in \Xi_\varepsilon^\pm \bigcup \partial\omega_\varepsilon; \qquad (2)$$

$$\partial u^\varepsilon / \partial n(x) = p(x), \qquad x \in \Gamma. \qquad (3)$$

Here, $\partial/\partial n$ stands for the derivative in the direction of the outward (with respect to Ω_ε) normal vector n. It supposed that a given traction force p satisfies the usual solvability condition.

After finding the solution u^ε of Eqs. (1)-(3) we shall consider the energy functional

$$I(u^\varepsilon;\Omega_\varepsilon) = 1/2 \int_{\Omega_\varepsilon} |\nabla_x u^\varepsilon(x)|^2 \, dx - \int_\Gamma p(x) u^\varepsilon(x) ds_x = -1/2 \int_\Gamma p(x) u^\varepsilon(x) ds_x. \quad (4)$$

We need asymptotic formulas of the solution u^{ε} as $\varepsilon \to 0$. In particular, its substitution into Eq. (4) gets the leading term in expansion of the energy functional, which represents the topological derivative caused by the diminishing hole ω_{ε} at the tip O of the crack Ξ_0 posed in the domain Ω.

2.2 The first limit problem

As $\varepsilon \to 0$ the hole ω_{ε} is collapsed to the point O, the boundary condition at the contour $\partial\omega_{\varepsilon}$ in Eq. (2) disappears, and Eqs. (1)-(3) form the limit (non-perturbed) problem with crack:

$$-\Delta_x v^0(x) = 0, \qquad x \in \Omega; \quad (5)$$
$$\partial v^0 / \partial n(x) = 0, \qquad x \in \Xi_0^{\pm}; \quad (6)$$
$$\partial v^0 / \partial n(x) = p(x), \qquad x \in \Gamma. \quad (7)$$

For the following use we introduce the polar coordinate system $r > 0, \varphi \in (0,2\pi)$ at the origin O. Applying the method of separation of variables for Eqs. (5)-(7), the asymptotic formula holds:

$$v^0(x) = \sum_{m=0}^{N} c_m^0 r^{m/2} \Phi^m(\varphi) + O(r^{(N+1)/2}), \quad r \to 0. \quad (8)$$

The coefficient c_1^0 in Eq. (8) is called the stress intensity factor. The angular functions:

$$\Phi^{2k}(\varphi) = \cos(k\varphi), \quad \Phi^{2k+1}(\varphi) = (2\pi)^{-1/2}(k + 1/2)\sin((k+1/2)\varphi), \quad k = 0,1,\ldots$$

are normalized as adopted in the fracture mechanics. The other functions in the Fourier series:

$$\Psi^0(\varphi) = -(2\pi)^{-1}, \quad \Psi^{2k}(\varphi) = (2k\pi)^{-1}\cos(k\varphi), \quad \Psi^{2k-1}(\varphi) = (2\pi)^{-1/2}(k-1/2)^{-2}\sin((k-1/2)\varphi)$$

for $k = 1,2,\ldots$ are normalized to satisfy the following conditions:

$$q(X^m, Y^n; S_R(O)) = \delta_{mn}, \quad Y^0 = (\log r)\Psi^0, \quad Y^n = r^{-n/2}\Psi^n, \quad X^m = r^{m/2}\Phi^m. \quad (9)$$

Here $S_R(O)$ stands for the circle of radius R centred at O, and q denotes the integral

$$q(U,V;\partial G) = \int_{\partial G} \{\partial U/\partial n(x) V(x) - U(x)\partial V/\partial n(x)\} ds_x = \int_G \{V(x)\Delta U(x) - U(x)\Delta V(x)\} dx. \quad (10)$$

The weight functions $\varsigma^m(x)$ are to be found as the non-energetic singular solutions of the homogeneous problem in Eqs. (5)-(7) when $p = 0$. They enjoy the following asymptotic behaviour:

$$\varsigma^m(x) = r^{-m/2}\Psi^m(\varphi) + O(1), \quad r \to 0.$$

With their help, the coefficients in Eq. (8) can be determined by the formulas:

$$c_m^0 = \int_{\Gamma} p(x)\varsigma^m(x) ds_x, \quad m = 0,1,\ldots$$

2.3 Special solutions

In the domain $R^2 \setminus (\Xi_\infty \bigcup \overline{\omega})$ we consider the series of Neumann problems for $k = 1,2,\ldots$

$$-\Delta_\xi \eta^k(\xi) = 0, \qquad \xi \in R^2 \setminus (\Xi_\infty \bigcup \overline{\omega}); \quad (11)$$

$$\partial_\nu \eta^k(\xi) = 0, \qquad \xi \in (\Xi_\infty^\pm \setminus \overline{\omega}) \bigcup \partial\omega; \quad (12)$$

$$\eta^k(\xi) = X^k(\xi) + o(1), \qquad |\xi| \to \infty. \quad (13)$$

Here ∂_ν denotes derivative in the direction of inward (with respect to ω) normal vector ν. Consider the coefficient $m_\omega^{1,1}$ in the asymptotic expansion of the special solution η^1:

$$\eta^1(\xi) = \rho^{1/2}\Phi^1(\varphi) + m_\omega^{1,1}\rho^{-1/2}\Psi^1(\varphi) + O(\rho^{-1}), \quad \rho \to \infty. \quad (14)$$

Using Green's formula and the normalization conditions in Eq. (9) we obtain

$$m_\omega^{1,1} = \int_{R^2 \setminus (\Xi_\infty \bigcup \overline{\omega})} \left|\nabla_\xi (\eta^1 - X^1)\right|^2 d\xi - \int_{\omega \setminus \Xi_\infty} \left|\nabla_\xi X^1\right|^2 d\xi. \quad (15)$$

We consider the case when $\omega = \{r \in (0,l), \varphi = \beta\}$. It describes the crack $\Xi_\infty \bigcup \omega$ kinking at the origin O with the fixed angle $\beta \neq 0$ to the branch $\omega = (O,Q)$ of the length $l > 0$. In this case, the second term in Eq. (15) disappears, and similarly to Eq. (8) the expansion holds

$$\eta^1(\xi) = C_0^1 + C_1^1 \rho_\beta^{1/2} \Phi^1(\varphi_\beta) + O(\rho_\beta), \quad \rho_\beta \to 0 \quad (16)$$

written in the local polar coordinated $\rho_\beta > 0, \varphi_\beta \in (0,2\pi)$ around the crack tip Q. Here C_1^1 is the stress intensity factor of the special solution η^1 of Eqs. (11)-(13) at $k = 1$. Following the Nazarov's method and applying path-independent integrals we suggest the equality

$$q(\eta^1, \rho\partial_\rho \eta^1; S_R(O)) = q(\eta^1, \xi \cdot \nabla_\xi \eta^1; S_\varepsilon(Q)). \quad (17)$$

Substituting the expansions given in Eq. (14) and Eq. (16) into Eq. (17) and using Eq. (9), from Eq. (15) and Eq. (17) we calculate

$$\begin{aligned} m_\omega^{1,1} = \; & -q\Big(\rho^{1/2}\Phi^1(\varphi) + m_\omega^{1,1}\rho^{-1/2}\Psi^1(\varphi), 1/2\{\rho^{1/2}\Phi^1(\varphi) - m_\omega^{1,1}\rho^{-1/2}\Psi^1(\varphi)\}; S_R(O)\Big) \\ & = -q\Big(C_0^1 + C_1^1\rho_\beta^{1/2}\Phi^1(\varphi_\beta), -lC_1^1\rho_\beta^{1/2}\Phi^1(\varphi_\beta); S_\varepsilon(Q)\Big) = l(C_1^1)^2. \end{aligned} \quad (18)$$

2.4 The second limit problem

Based on the above results, now we consider the perturbed problem in Eqs. (1)-(3). The asymptotic formula for the solution u^ε is well known; see, e.g., *(10)*. The method of matched asymptotic expansions (see *(7, 14)*) implies the following structure for the inner asymptotic representation:

$$u^{\varepsilon}(x) \approx c_0^0 + c_1^0 \varepsilon^{1/2} \eta^1(\xi), \qquad (19)$$

which is valid in a small neighbourhood of O. Here we used the stretched variables $\xi = \varepsilon^{-1} x$. On the other hand, around the tip Q of the kinked crack we have the following asymptotic expansion

$$u^{\varepsilon}(x) \approx c_0^{\varepsilon} + c_1^{\varepsilon} r_{\varepsilon}^{-1/2} \Phi^1(\varphi_{\varepsilon}), \quad r_{\varepsilon} \to 0. \quad (20)$$

From Eq. (16), Eq. (19), and Eq. (20) we infer that $c_1^{\varepsilon} = c_1^0 C_1^1 + O(\varepsilon^{1/2})$.

2.5 Expansion of the energy functional

Away from some neighbourhood of O the outer asymptotic expansion has the form

$$u^{\varepsilon}(x) \approx v^0(x) + \varepsilon (c_1^0 m_{\omega}^{1,1}) \varsigma^1(x). \qquad (21)$$

Substituting Eq. (21) into the energy functional in Eq. (4) gets due to Eq. (18)

$$\begin{aligned} I(u^{\varepsilon};\Omega_{\varepsilon}) &= I(v^0;\Omega) - \varepsilon/2\,(c_1^0 m_{\omega}^{1,1}) \int_{\Gamma} p(x) \varsigma^1(x) ds_x + O(\varepsilon^{3/2}) \\ &= I(v^0;\Omega) - \varepsilon/2\,(c_1^0)^2 m_{\omega}^{1,1} + O(\varepsilon^{3/2}) \qquad (22) \\ &= I(v^0;\Omega) - (\varepsilon l)(c_1^0 C_1^1)^2/2 + O(\varepsilon^{3/2}). \end{aligned}$$

Note that due to the expansion in the second limit problem, Eq. (22) yields the well-known formula

$$I(u^{\varepsilon};\Omega_{\varepsilon}) = I(v^0;\Omega) - (\varepsilon l)(c_1^{\varepsilon})^2/2 + O(\varepsilon^{3/2}).$$

Thus, from Eq. (22) we conclude with the topological derivative, which is of the first order,

$$\lim_{\varepsilon l \to 0} \frac{I(u^{\varepsilon};\Omega_{\varepsilon}) - I(v^0;\Omega)}{\varepsilon l} = (c_1^0 C_1^1)^2/2 \quad (23)$$

with respect to the diminishing crack branch ω_{ε} of the length εl.

3. CONCLUSION

From the practical point of view, we obtained the asymptotic expansion of the potential energy expressed in the terms of stress intensity factors, which are of the primary importance for engineers. We observe that the topological derivative in Eq. (23) is expressed with the help of two stress intensity factors. While the former (initial) stress intensity factor c_1^0 depends on the specific choice of physical parameters and geometric data of the reference crack problem before kink, which is given in Eqs. (5)-(7), the latter C_1^1 is the universal parameter depending on the geometry of the kinked infinite domain only. This implicit quantity is to be determined from the special solution η^1 for model problem of the crack kinking from Eqs. (11)-(13) at $k = 1$.

In the particular case, when the reference state is smooth, thus $c_1^0 = 0$, then the first-order term is zero, and the topological derivative is determined by the higher-order term in the asymptotic expansion in Eq. (22). This fact agrees with the smooth conception of topological derivatives.

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