Fair Division*

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Abstract This survey considers approaches to fair division from diverse disciplines, including mathematics, operations research and economics, all of which place fair division within the same general structure. There is some divisible item, or some set of items (which may be divisible or indivisible) over which a set of agents has preferences in the form of value functions, preference rankings, etc. Emphasizing the algorithmic part of the literature, we will look at procedures for fair division, and from an axiomatic point of view will investigate important properties that might be satisfied or violated by such procedures. Links to the analysis of aggregating individual preferences (Nurmi, Chapter 10 of this Handbook) and bargaining theory (Kibris, Chapter 9) will become apparent. Strategic and computational aspects are also considered briefly. In mathematics, the fair division literature focuses on cake-cutting algorithms. We will provide an overview of the general framework, discuss some specific problems that have been studied in depth, such as pie-cutting and cake-cutting, and present several procedures. Then, moving to an informationally scarcer framework, we will consider the problem of sharing costs or benefits. Based on an example from the Talmud, various procedures to divide a fixed resource among a set of agents with different claims will be discussed. Certain changes to the framework, e.g. the division of costs or variable resources, will then be investigated. The survey concludes with a short discussion of fair division issues from the viewpoint of economics.

1 Introduction

What do problems of cutting cakes, dividing land, sharing time, allocating costs, voting, devising tax-schemes, evaluating economic equilibria, etc. have in common? They are all concerned - in one form or the other - with **fairness**. This, of course, needs some idea about what fairness actually means. The literature is full with different approaches to the challenges of fair division and allocation problems. Moreover, those approaches come from many - quite different - scientific areas. Certainly, one immediate first thought when thinking about fairness is to tackle it from a philosophical point of view. However, purely philosophical issues are left aside in this survey and the reader is referred to two extensive coverages by Kolm [42] and Roemer [61]. Applications of fairness principles to peace negotiations can be found in Chapter 6 of this volume (Albin and Druckman [1]). A second well established link to fairness comes from the literature on bargaining theory and cooperative game theory discussed in Chapter 9 of this volume (Kibris [40]).

In contrast to cooperative game theory, in this survey we want to focus mostly on the algorithmic (or procedural) aspect of fair division. In particular we will investigate approaches to fairness in division and allocation problems discussed predominantly in the disciplines of **Mathematics**, **Operations Research** and **Economics**.

Of course, one could still think of many other seperate areas that have more or less close links to fairness issues, some of them being based on similar models as discussed in the following. Examples are the apportionment theory (see Balinski and Young [4]), the voting power literature (see Felsental and Machover [31]), the literature on voting

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systems (see Chapter 10 of this volume, Nurmi [55]) or the analysis of equal opportunities (see Pattanaik and Xu [57]). Experimental treatments of fair division aspects have also been undertaken (see e.g. Fehr and Schmidt [29]).

Fairness, as discussed and analysed through the above three approaches, has been a booming area in recent years with numerous papers looking at various different aspects. Extensive surveys and books have been written, mostly focusing on one of the above approaches in detail, e.g. Brams and Taylor [20] covering the discipline of mathematics, Moulin [47] approaches in connection to operations research and Thomson [74] fairness models in economics.[2]

In general, all our approaches are concerned with mappings, assigning to each division problem a (if possible single-valued) solution in the form of a division or allocation. Domain and codomain of such mappings will differ w.r.t. which approach is going to be used, mostly depending on structural differences based on answers to the following broad questions:

1. **What is to be divided?**
   One of the main components of a fair division problem is the object that is going to be divided. Such objects range from the $[0, 1]$-interval, that needs to be partitioned e.g. in classical cake-cutting examples, over sets of (indivisible) items, to real numbers, representing costs to be divided in cost-sharing problems. Extensions and/or restrictions to the above in the form of e.g. money, cost functions or network structures do come up frequently.
2. **How are agents' preferences represented?**
   Another major input in our fairness models are the agents' valuations of what is going to be divided, or simply their preferences. Those can be represented in various different forms, depending on how much information seems to be acceptable in the division problem. Preference representation ranges from cardinal value functions or measures, via ordinal preferences or rankings of items, to simple claims of agents somehow representing their ideal points.
3. **How are we dividing? What do we want to achieve?**
   Based on what is to be divided and the agents' preferences, the main goal now is to determine fair division procedures or algorithms. From an algorithmic point of view, we need to specify the rules of the procedure. Which agent can do what? Is a referee needed? What are the informational and/or computational requirements? And besides all that we need to know whether such a procedure leads to a fair division. This latter question will usually be answered on the basis of certain axioms or properties satisfied by an allocation that somehow represent the idea of fairness. E.g. one of the major axioms in that respect is **envy-freeness**, i.e. a division/allocation such that each agent is at least as well off with her own share than with any other agent's share. Of course, depending on the context many other axioms play a role in the fair division literature. Actually a large part of the literature is predominantly concerned with the axiomatic part of fair division, i.e. with existence or characterization results.

This survey aims at providing answers to the above questions depending upon the discipline in which one operates. For each approach we specify the usual framework and state a few of the main results and/or present and discuss the qualities of fair division procedures. All this will be accompanied by numerical examples. Links between different areas will be established and some open questions raised. The focus will lie on cake-cutting, the division of indivisible objects and cost allocation. For those situations we will pick some procedures that will be explained and discussed in detail to give a feeling of how certain fair-division issues have recently been handled in the literature.

2 Cutting Cakes and Dividing Sets of Items

In Hesiod's Theogeny, which dates back about 2800 years, the Greek gods Prometheus and Zeus were arguing over how to divide an ox. Eventually the division was that Prometheus divided the ox into two piles and Zeus chose one (see Brams and Taylor [20]). This can be taken as the standard example in the fair division literature where most effort has been devoted to by mathematicians, with the simple exception that meat has been substituted by cake and therefore it is usually called **cake-cutting**. The "cake" stands as a metaphor for a single heterogeneous good, and the goal is to divide the cake between some agents (often called players). However, other situations such as the division of various divisible and/or indivisible items have also been investigated. Inheritance problems or divorce settlements can be seen as immediate examples for such situations.

[2] See also Young [80] for an excellent book-length treatment of various fair division aspects.

2.1 One divisible object

The focus in cake-cutting was - for a long time - mostly on algorithms or procedures, following the work of Polish mathematicians in the 1940s. Many - still widely discussed - cake-cutting procedures date back to Hugo Steinhaus and his contemporaries, e.g. Steinhaus' "lone-divider procedure" and the "last-diminisher procedure" by Stefan Banach and Bronislaw Knaster (Steinhaus [64]). Brams and Taylor [20] provide a detailed discussion of those and other procedures and give a historical introduction.[3]

2.1.1 Formal Framework

More formally, this approach to fair division is mostly concerned with dividing some set C (the "cake") over which various players have (different) preferences. Usually C is just the one-dimensional $[0,1]$-interval. The goal is to find an allocation of disjoint subsets of C for n players (mostly in form of a partition of C). Mathematically, as we need to value subsets of C, we use a σ-algebra on C, i.e. a collection of subsets $\mathscr{W}$ of C with the properties that C is in $\mathscr{W}$, $S \in \mathscr{W}$ implies $C \backslash S \in \mathscr{W}$, and that the union of countably many sets in $\mathscr{W}$ is also in $\mathscr{W}$. The pair $(C, \mathscr{W})$ is called a measurable space.[4]

Now, given such a measurable space $(C, \mathscr{W})$, agents' preferences are represented by (probability) measures (also called value functions) on $\mathscr{W}$, i.e. $\mu : \mathscr{W} \rightarrow [0,1]$, with $\mu(\emptyset) = 0$, $\mu(C) = 1$ and if $S_1, S_2, \ldots$ is a countable collection of pairwise disjoint elements of $\mathscr{W}$, then $\mu(\bigcup_{i=1}^{\infty} S_i) = \sum_{i=1}^{\infty} \mu(S_i)$, i.e. μ is countably additive.

Mostly, μ is assumed to be non-atomic, i.e. for any $S \in \mathscr{W}$, if $\mu(S) > 0$ then for some $T \subseteq S$ it follows that $T \in \mathscr{W}$ and $\mu(S) > \mu(T) > 0$. The non-atomicity condition is widely used in the cake-cutting literature, and without it a fair division might not be possible. Also, often a weaker version of countable additivity, namely finite additivity, sufficies, i.e. for all disjoint $S, T \in \mathscr{W}$, $\mu(S \cup T) = \mu(S) + \mu(T)$. This weaker condition does, however, preclude any procedure using an infinite number of cuts. Finally, a widely used condition is concerned with the possibility of certain players attaching positive values to pieces whereas others attach no value to the same piece. More precisely, a measure μ_i is *absolutely continuous* with respect to measure μ_j if and only if for all $S \subseteq C$, $\mu_j(S) = 0 \Rightarrow \mu_i(S) = 0$. A strengthening of this condition is *mutual absolute continuity* saying that for any $S \subseteq C$, if for some j, $\mu_j(S) = 0$, then $\mu_i(S) = 0$ also for all $i \neq j$.

2.1.2 Properties

Some of the earliest theoretical results on which later cake-cutting results are based, are Lyapunov's Theorem [43] and results by Dvoretsky, Wald and Wolfovitz [27] (see also Barbanel [5]). Those have widely been used, some applications can be found in Barbanel and Zwicker [7]. In particular, they show the following:

Proposition 1. *For any $(p_1, p_2, \ldots, p_n) \in \Re^n_+$ such that $\sum_{i \in N} p_i = 1$, there is a partition $(S_1, \ldots, S_n)$ of C such that for all $i, j = 1, 2, \ldots, n$, $\mu_i(S_j) = p_j$*

Proposition 1 immediately tells us, that there always exists an allocation such that every agent receives a piece (i.e. a set of subsets of C) he or she values at $\frac{1}{n}$ and everybody else values at $\frac{1}{n}$. Beware though, that there is no mentioning of whether a player gets one connected piece or many small disconnected crumbs.

Having established a first idea about what is going to be divided and what preferences tend to look like, we can now discuss what this literature wants to achieve. In general, the focus is on procedures and the satisfaction of certain properties by the allocations that those procedures select. Those properties - at least to some extent - provide an idea about what "fairness" could mean. A small selection of such properties is the following:[5]

Definition 1. Let $P = (S_1, \ldots, S_n)$ be a partition of C, then P is

- **proportional** if and only if for all $i \in N$, $\mu_i(S_i) \geq \frac{1}{n}$

[3] A brief survey over some parts ot the mathematics literature on fair division has recently been provided by Brams [13].

[4] See Weller [77] for a general approach to fair division of measurable spaces.

[5] Those are the properties most often used in the literature. However, there do exist many other properties in this literature, e.g. strengthenings or weakenings of the above properties (see e.g. Barbanel [5]).

- **envy-free** if and only if for all $i, j \in N$, $\mu_i(S_i) \geq \mu_i(S_j)$
- **equitable** if and only if for all $i, j \in N$, $\mu_i(S_i) = \mu_j(S_j)$
- **efficient** if and only if there is no partition $P' \neq P$ s.t. for all $i \in N$, $\mu_i(S'_i) \geq \mu_i(S_i)$ and $\mu_j(S'_j) > \mu_j(S_j)$ for some $j \in N$.

A proportional division gives each agent a piece that she values at least $1/n$ of the cake. Envy-freeness requires that no agent would prefer the piece of another agent. If all agents attach the same valuation to their pieces relative to the whole cake, a division is called equitable. Efficiency is defined in its usual way.

2.1.3 Cake-Cutting Procedures

Algorithms or cake-cutting procedures give instructions on how to cut the cake, i.e. what partition to select, so that certain properties are satisfied by the allocation. Formally, let a procedure be denoted by ϕ, assigning to any division problem (N, C, μ), with $\mu = (\mu_1, ..., \mu_n)$, a partition of C.

Procedures can be distinguished on the basis of certain technical aspects. One distinction is between discrete and moving-knife procedures. In **discrete procedures**, the players' moves are in a sequence of steps, whereas in **moving-knife procedures**, there is a continuous evaluation of pieces of cakes by the single players. Intuitively, the latter works by moving a knife along a cake, asking the players to constantly evaluate the pieces to the left and to the right of the knife.

A further essential distinction is based on the *number of (non-intersecting) cuts* allowed for partitioning the cake. Procedures using the minimal number of cuts, namely $n-1$, assign to each agent a connected piece. In certain situations this might be a plausible - if not compelling - requirement. Sometimes, especially when there is a larger number of players, using the minimal number of cuts drastically restricts the properties that can be satisfied.

In 2-player division problems, probably the most widely known one-cut procedure (which is the minimal number of cuts) is **cut and choose** as used in the Greek mythology by Prometheus and Zeus. In this procedure, one agent - the "cutter" - cuts the cake into two pieces and the other agent - the "chooser" - takes one of the two pieces leaving the cutter with the other piece. Obviously, in the absence of any information about the chooser's value function, if the cutter cuts the cake into two pieces that she values the same, the final allocation will satisfy most of the above properties, given that the chooser takes a piece that is at least as large as the other piece. The final allocation is proportional as both agents get at least a value of $\frac{1}{2}$ in their eyes. In two-agent settings this implies envy-freeness, and - assuming the value functions being mutually absolutely continuous - efficiency. The only property that is violated is equitability. Let us illustrate this in an example:

Example 1. Let $C = [0, 1]$ denote a cake whose left half is made of chocolate and whose right half is made of vanilla. $N = \{1, 2\}$ and the players' values for any subinterval S of C are given by $\mu_i(S) = \int_S f_i(x)dx$ where:

$$f_1(x) = 1,\ f_2(x) = \begin{cases} \frac{4}{3} \text{ if } x \in [0, \frac{1}{2}] \\ \frac{2}{3} \text{ if } x \in (\frac{1}{2}, 1] \end{cases}$$

Hence player 1 is indifferent between chocolate and vanilla, whereas player 2 values chocolate twice as much as vanilla. In a no-information setting, player 1 - as cutter - can only guarantee a 50%-share of the cake to himself if he cuts the cake at a point where both pieces are of equal value to him[6], i.e. at point $\frac{1}{2}$. Now, player 2 - the chooser - chooses the left (i.e. the chocolate) piece leaving player 1 with the right (i.e. the vanilla) piece. This allocation gives player 1 a piece he values at exactly 50% of the total cake and player 2 a piece she values 67% of the total cake. Which of the discussed properties are satisfied? Each player values his or her own piece at least as much as the other player's piece, hence the allocation is envy-free, and this implies proportionality. As both players attach positive value to the whole cake (i.e. to the the whole interval, and therefore we have mutual absolute continuity satisfied), the allocation is efficient among 1-cut allocations on such intervals. Finally, as player 2 attaches more value to her piece (relative to the whole cake) than player 1 attaches to his piece, the allocation violates equitability.

Although the existence of an envy-free allocation for any number of agents can be shown (recall proposition 1), little is known about procedures that lead to such an envy-free allocation.[7] The extension of "cut and choose" to 3

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[6] This is like following the maximin solution concept as used in non-cooperative game theory.

[7] Su [66] uses Sperner's lemma to show the existence of an envy-free cake division under the assumption that the players prefer a piece with mass to no piece (i.e. players being "hungry") and preference sets being closed.

or more players using 2 cuts by a discrete procedure is difficult. Already for $n = 3$, not even proportionality can be guaranteed. The only guarantee that can be given is that each player gets a piece that she or he values at least $\frac{1}{4}$ (see Robertson and Webb [60]).

For 3 players, the discrete procedure guaranteeing envy-freeness with the fewest cuts - namely at most 5 - has been independently discovered by John Selfridge and John Conway in the 1960s (but never published - hence see Brams and Taylor [20] for a discussion). For 4 players there is no discrete procedure that uses a bounded number of cuts. Envy-freeness for 3 agents with the minimal number of 2 cuts, is only achieved by two moving-knife algorithms devised by Stromquist [65] and Barbanel and Brams [6]. The latter also provide a 4 player moving-knife procedure using 5 cuts (and various moving knifes). Fewer moving knifes (which could be considered "easier"), but more cuts (namely 11), is what has been achieved by the procedure in Brams, Taylor and Zwicker [22]. Possible extensions (to more players) and simplifications (to fewer cuts and/or fewer moving knifes) are still open.[8]

When increasing the number of players, often proportionality is the most one can hope for. One considerably attractive discrete procedure guaranteeing a proportional allocation for any number of players is **divide and conquer** (Even and Paz [28]). It works by asking the players successively to place marks on a cake that divide it into equal or approximately equal halves, then halves of these halves, and so on. Interestingly, it turns out that divide and conquer minimizes the maximum amount of individual envy[9] among all discrete procedures and fares no worse w.r.t. the total amount of envy compared to other discrete procedures (see Brams et al. [17]).

2.1.4 Entitlements

Things slightly change whenever the entitlements are not the same for all players, i.e. say one player is entitled to twice as much as the other player, and hence the cake needs to be divided accordingly. An analysis of such situations has been provided by Brams et al. [16]. Non-equal entitlements require a redefinition of well-known properties. Given a vector of entitlements $(p_1, ..., p_n)$, s.t. $p_i > 0$ and $\sum_{i=1}^n p_i = 1$, an allocation $P = (S_1, ..., S_n)$ is **proportional** if $\frac{\mu_i(S_i)}{p_i} = \frac{\mu_j(S_j)}{p_j}$ for all $i, j \in N$, i.e. each player gets the same amount according to the proportions in the vector of entitlements.[10] An allocation P is **envy-free** if $\frac{\mu_i(S_i)}{p_i} \geq \frac{\mu_i(S_j)}{p_j}$ for all $i, j \in N$, i.e. no player thinks another player received a disproportionally large piece, based on the latter player's entitlement. As a final property we say that an allocation is **acceptable** if each player receives a piece valued at least as much as her entitlement, i.e. $\mu_i(S_i) \geq p_i$ for all $i \in N$.

It turns out, that even in 1-cut cake-cutting problems with $n = 2$, for some individual preferences the allocation may not assign acceptable pieces (although the allocation might be proportional). This is in contrast to the fact that there is an envy-free and efficient allocation whenever there are equal entitlements and can be seen in the following example taken from Brams et al. [16]:

Example 2. Let $C = [0,1]$, $N = \{1,2\}$ and the players' values be given by $\mu_i(S) = \int_S f_i(x)dx$ where

$$f_1(x) = 1 \text{ and } f_2(x) = \begin{cases} 4x & \text{if } x \in [0, \frac{1}{2}] \\ 4-4x & \text{if } x \in (\frac{1}{2}, 1] \end{cases}$$

Assume that the players are entitled to unequal portions, namely p and $1-p$ for $\frac{1}{2} < p < 1$. If the cake is cut at $x = p$, then player 1 gets piece $[0, p]$ which he values at $\mu_1([0, p]) = p$. Player 2 gets the remainder $(p, 1]$ which she values at $\mu_2((p, 1]) = \int_p^1 (4-4x)dx = 2(1-p)^2$. As $2(1-p)^2 < 1-p$ for $p > \frac{1}{2}$, player 2 receives less than its entitled share $1-p$. As we can use the same argument for $0 < p < \frac{1}{2}$, we see that there is no acceptable allocation from a single cut given those entitlements and value functions. There is, however, a proportional allocation possible by solving for the cut point x in $x : 2(1-x)^2 = p : 1-p$ in which both players receive pieces that they value less than their entitlements.

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[8] Besides cutting cakes, similar algorithms are used to divide chores, i.e. items that are considered undesirable. Su [66] guarantees an ε-approximate envy-free solution, Peterson and Su [59] develop a simple and bounded procedure for envy-free chore division among 4 players.

[9] The amount of individual envy of a player is determined by the number of other players she envies.

[10] This could also be seen as the equitability property for unequal entitlements.

2.1.5 Cakes and Pies

Another distinction can be made between cutting cakes and pies. Cakes are represented as closed intervals, pies are infinitely divisible, heterogeneous and atomless one-dimensional continuums whose endpoints are topologically identified, such as a circle (Thomson [73]). If we remain in such a one-dimensional setting, the minimal number of cuts necessary to cut a pie into n pieces is n. Gale [35] was probably the first to suggest a difference between cakes and pies. His question of whether for n players there is always an envy-free and efficient allocation of a pie using the minimal number of cuts, was answered in the affirmative for $n = 2$ by Thomson [73] and Barbanel et al. [8]. The latter provided the following two results: First, if players' measures are not mutually absolutely continuous, an envy-free and efficient allocation of a cake may be impossible. Second, there exist players' measures on a pie for which no partition of a pie is envy-free and efficient (regardless of the assumption about the absolute continuity of the players' measures w.r.t. each other).

To prove the first statement, Barbanel et al. [8] use the following example: Let $|N| = 3$ and the players' measures be uniformly distributed over the three intervals as stated in Table 1.

Table 1 Players' measures

	$[0,\frac{1}{6}]$	$[\frac{1}{6},\frac{1}{3}]$	$[\frac{1}{3},1]$
Player 1	$\frac{1}{3}$	0	$\frac{2}{3}$
Player 2	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
Player 3	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$

Because player 1 places no value on $[\frac{1}{6},\frac{1}{3}]$, the measures are not mutually absolutely continuous. Barbanel et al. [8] show that any allocation $P = (S_1, S_2, S_3)$ cannot be both, envy-free and efficient. To be envy free, $\mu_i(S_i) \geq \frac{1}{3}$ for all i. If player 1 receives the leftmost piece $[0,x]$, then for $x > \frac{1}{3}$, there is not enough cake left to give at least $\frac{1}{3}$ to the other two players. If $x < \frac{1}{3}$, then players 2 and 3 need to divide the remainder equally so that they do not envy each other. But if so, player 1 will envy the player who gets the rightmost piece. If $x = \frac{1}{3}$, then the allocation that assigns $[0,\frac{1}{3}]$ to player 1, $(\frac{1}{3},\frac{2}{3}]$ to player 2 and $(\frac{2}{3},1]$ to player 3, is envy free, but not efficient as it is dominated by the allocation $([0,\frac{1}{6}],(\frac{1}{6},\frac{7}{12}],(\frac{7}{12},1])$, that gives larger value to players 2 and 3 and the same value as before to player 1. For any other player getting the leftmost piece, we see again that envy-freeness requires pieces to be $[0,\frac{1}{3}]$, $(\frac{1}{3},\frac{2}{3}]$ and $(\frac{2}{3},1]$, but this can be dominated by the allocation stated before.

Barbanel et al. [8] also show for a pie that for certain players' measures there is no allocation that is envy-free and equitable if there are four or more players. As we can always find such allocations for two players, this leaves the case for three players as an open question.

If we refer to the previous example 2 where we used entitlements, it is interesting, that in a two-player pie-cutting problem, assigining acceptable pieces is always possible (see Brams et al. [16]). This is due to the second cut necessary in cutting pies into two pieces. However, an increase from two to three players may rule out proportional allocations with unequal entitlements using three cuts.

There are also geometric approaches to cake- and pie-cutting. Barbanel [5] uses geometry to analyse existence results that deal with fairness. Thomson [73] develops a geometric representation of feasible allocations and of preferences in two-dimensional Euclidean space, and (re)proves various results in this geometric framework.

2.1.6 Incentives

Incentives do play an important role in fair division in the sense that one wants to know whether a division procedure can be manipulated by players not announcing their true preferences. In the cake-cutting literature, a procedure is considered *truth-inducing* if it guarantees players at least a $\frac{1}{n}$ share of the cake if and only if they are truthful.[11] Players that are sufficiently risk-averse, therefore, have good reason not to lie about their preferences in such a procedure.[12]

[11] Other thresholds besides $\frac{1}{n}$ could be used, especially if we have a situation in which the players have unequal entitlements.

[12] A certain similarity to maximin behavior can be observed (see also Crawford [24] for the use of maximin behavior in economic models).

Based on such a concept of non-manipulability, there exist various results about truth-inducing procedures (e.g. Brams et al. [16] [17]).

The common - and generally more standard - approach (e.g. in Thomson [73]) is based on the following definition: A procedure ϕ is strategy-proof if for all profiles of value functions μ, each i and all μ_i', $\mu_i(\phi_i(N,C,\mu)) \geq \mu_i(\phi_i(N,C,(\mu_i',\mu_{-i})))$. I.e. no agent is ever allowed to have an incentive to misrepresent her preferences, i.e. truth-telling should be a dominant strategy. This is a much stronger property widely used in economics, game theory and social choice theory, leading to mostly negative results.

Using a slight weakening of this incentive requirement, Thomson [72] provides a procedure - **divide and permute** - that fully implements in Nash equilibrium the no-envy solution in n-person fair division problems. **Cut and choose** - for $n = 2$ - gives only a partial implementation of the envy-free solution. Only the envy-free allocation most favorable to the divider is obtained in equilibrium (beware that players are fully informed about each others preferences and therefore the divider has an advantage compared to the no-information case). Thomson [72] obtains a full implementation in the sense of obtaining all envy-free allocations.

Nicolo and Yu [54] follow a similar approach for a two-player fair division problem to implement an envy-free and efficient solution in subgame perfect equilibrium. Their **strategic divide and choose** procedure tries to combine allocational aspects and procedural aspects in fair division problems.[13]

2.1.7 Computational Aspects

An interesting aspect of such algorithms is their complexity (see Woeginger and Sgall [78]). The complexity of a cake cutting procedure is generally measured by the number of cuts (usually including the informational queries in the process) performed in the worst case. As proportionality is the best we can guarantee w.r.t. fairness for $n > 3$, it only makes sense to look at complexity w.r.t. proportional procedures. The best deterministic procedure so far is divide and conquer which uses $O(n \log n)$ cuts, however, Even and Paz [28] design a randomized protocol that uses an expected number of $O(n)$ cuts. There are still open questions in whether those numbers can be improved upon (see Woeginger and Sgall [78]).

2.2 Allocating divisible and/or indivisible objects

In case there is not one heterogeneous divisible item, but various (in)divisible items (e.g. different items in a divorce settlement, components of a contract between a firm and its employees, etc.), the formal framework changes in the sense that the "cake" C contains a finite number of items. Depending on the context, various (restrictive) assumptions on preferences are assumed. One such assumption is that the value of items is independent of each other, i.e. there are no complementarities and/or synergies between the items. This is often necessary to allow using a ranking of the items in C to say something about the value of subsets of C and implies additive utilities of the items.[14] Otherwise, a ranking of all possible subsets would be necessary, making this a computationally difficult problem. Based on individual rankings on a set of indivisible items, Brams, Edelman and Fishburn [14] provide a whole set of paradoxes, showing the difficulites of getting fair shares for everybody (see also Brams et al. [15]).

One procedure taking explicit care of such situations is *Adjusted Winner* introduced by Brams and Taylor [20] (see also their book-length popular treatment [21]).[15] Using their procedure, they always determine a fair allocation s.t. at

[13] E.g. the simple divide and choose method leads to an allocation that is both envy-free and efficient (for $n-1$ cut procedures). However, for non-identical preferences (actually, non-equivalent 50-50 points, i.e. the point which divides the cake into two pieces of exactly the same value for that player), whoever is the divider will envy the chooser for being in the position of receiving a value of more than 50% of the cake whereas the divider can only guarantee a value of 50% to herself. The fairness problem involved in that has been discussed e.g. by Crawford [23].

[14] For a detailed discussion of ranking sets of items based on a ranking of the items see Barbera et al. [9].

[15] The procedure has a certain similarity to the use of the greedy algorithm in knapsack problems. See Kellerer et al. [39] for an extensive treatment of knapsack problems.

most one item needs to be divided.[16] In principle, the formal framework is identical to the cake-cutting situation if one thinks of the items being arranged one next to each other (see Jones [37]).

A very challenging issue is the allocation of indivisible goods with no divisible items (such as money) involved. One recent procedure to assign items to players in that respect is the *undercut procedure* (Brams et al. [19]) which will now be presented in more detail. Let C be the set of m indivisible items and two players, 1 and 2, be able to rank those items from best to worst. Players' preferences on C are additive, i.e. there are no complementarities between the items. Before defining the procedure, a few definitions are required:

Definition 2. Consider two subsets $S,T \subseteq C$. We say that T is *ordinally less* than S if $T \subset S$ or if T can be obtained from S, or a proper subset of S, by replacing items originally in S by equally many lower-ranked items.[17]

Definition 3. A player regards a subset S as *worth at least 50%* if he or she finds S at least as good as its complement $-S$.

Definition 4. A player regards a subset S as a *minimal bundle* if (i) S is worth at least 50%, and (ii) any subset T of C that is ordinally less than S is worth less than 50%.

The rules of the undercut procedure (UP) are as follows:

1. Players 1 and 2 independently name their most-preferred items. If they name different items, each player receives the item he or she names. If they name the same item, this item goes into a so-called *contested pile*.
2. This process is repeated for every position in the players' rankings until all the items are either allocated to player 1, player 2 or to the contested pile.
3. If the contested pile is empty, the procedure ends. Otherwise, both players identify all of their minimal bundles of items in the contested pile and provide a list of those bundles to a referee.
4. If both players have exactly the same minimal bundles, there is no envy-free allocation of the contested pile unless they consider a bundle S and its complement $-S$ as minimal bundles, in which case we give S to one player and $-S$ to the other player. Otherwise the procedure ends without a division of the contested pile.
5. If both players' minimal bundles are not the same, the players order their minimal bundles from most to least preferred. A player (say 1) is chosen at random and announces his top ranked minimal bundle. If this is also a minimal bundle for player 2, then the top-ranked minimal bundle of player 2 is considered. Eventually one minimal bundle of a player will not be a minimal bundle of the other player. It becomes the proposal.
6. Assume that the proposal comes from player 1. Then player 2 may respond by (i) accepting the complement of player 1's proposed minimal bundle (which will happen if it is worth at least 50 percent to her) or (ii) *undercutting* player 1's proposal, i.e. taking for herself any subset that is ordinally less than player 1's proposal, in which case the complement of player 2's subset is assigned to player 1. The procedure ends.

Given steps 1 and 2, it is obvious that items in the contested pile are ranked the same by the two players (but of course they need not have the same positions in the original ranking of all items before undertaking steps 1 and 2). Now, Brams et al. [19] show that there is a nontrivial envy-free split[18] of the contested pile if and only if one player has a minimal bundle that is not a minimal bundle of the other player. If so, then UP implements an envy-free split as illustrated in the following example:

Example 3. Let $CP = \{a,b,c,d,e\} \subseteq C$ be the constested pile such that both players rank item a above item b above item c, etc. Now consider that one of player 1's minimal bundle, $MB_1 = \{a,b\}$, is not a minimal bundle for player 2 and let one of 2's minimal bundles be $MB_2 = \{b,c,d,e\}$. If player 1 offers the division ab/cde, i.e. bundle $\{a,b\}$ to player 1 and bundle $\{c,d,e\}$ to player 2, then player 2 will reject this proposal because the set $\{c,d,e\}$ must be worth less than 50% given that $\{b,c,d,e\}$ was a minimal bundle for her. Hence player 2 will undercut by proposing bde/ac, i.e. bundle $\{a,c\}$ to herself and bundle $\{b,d,e\}$ to player 1. As $MB_2 = \{b,c,d,e\}$, the bundle $\{b,d,e\}$ must be worth less than 50% and therefore $\{a,c\}$ is worth more than 50% to player 2. As $MB_1 = \{a,b\}$, the bundle $\{a,c\}$ must be worth less than 50% to player 1 and hence $\{b,d,e\}$ is worth more than 50% to him. This guarantees an envy-free division.

[16] Some papers such as Alkan, Demange and Gale [2] and Tadenuma and Thomson [69] discuss the allocation of indivisible items when monetary compensations are possible (i.e. in the presence of an - infinitely divisible - item).

[17] See also Taylor and Zwicker [70].

[18] An envy-free split is trivial if each player values its subset at exactly 50 percent.

An interesting feature of UP is that it is truth-inducing, i.e. sincerety is the only strategy that guarantees a 50% share (in case the CP can be split). Suppose a minimal bundle for player 1 is $\{a\}$, but he proposes the split ab/cde. If $\{c,d,e\}$ is worth at least 50% to player 2, she will accept the proposal and player 1 is better off than having told the truth. However, if player 1 is undercut (because $\{c,d,e\}$ is less than 50% for player 2), then the split would have been bde/ac giving player 1 the bundle $\{b,d,e\}$ he values less than 50% (as $\{a\}$ was a minimal bundle, $\{b,c,d,e\}$ was less than 50% already).[19]

3 Sharing Costs or Benefits

The most prominent example in this area comes from the 2000 year old Babylonian Talmud and goes as follows: A man, who died, had three wives, each of them having a marriage contract. These contracts specified the claims that the women had on the whole estate. The first woman had a claim of 100, the second a claim of 200 and the third a claim of 300. Now, if the estate was not enough to meet all the claims, some division of the estate was necessary. The Talmud specifies such a division for various values of the estate as stated in Table 2.

Table 2 Example from the Talmud

estate	claims		
	100	200	300
100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
200	50	75	75
300	50	100	150

As can be seen from Table 2, the formal framework now slightly differs from the one in the previous section.[20] The simplest setting considers the division of a joint resource among some agents having certain claims on that resource. There are many situations that are structurally similar to the example in the Talmud and many other division problems from the Talmud are discussed (see e.g. O'Neill [56]). In particular, a huge part of the literature is concerned with bankruptcy problems in which an estate needs to be divided among agents having different claims (see Thomson [71] for a survey). It also includes the division of a cost that needs to be jointly covered by a group that have different responsibilities in creating the cost.[21] More elaborate structures are of course possible by introducing cost functions, networks, etc. Such structures will be discussed later.

3.1 Dividing a Fixed Resource or Cost

Formally, we are concerned with dividing a resource, i.e. some $r \in \Re_+$, given agents' claims $x = (x_1,...,x_n) \in \Re^n_+$. The goal is to fairly share a deficit (in the case of $r \leq \sum_{i\in N} x_i$) or a surplus (in the case of $r \geq \sum_{i\in N} x_i$). Hence a fair division problem can be seen as a triple (N,r,x). A solution to such a fair division problem is then a vector of individual shares $y = (y_1,...,y_n) \in \Re^n_+$ s.t. $\sum_{i\in N} y_i = r$. A solution method (or rule, or procedure) ϕ assigns to each fair division problem (N,r,x) a solution $\phi(N,r,x) = y$.

[19] A closely linked problem is the housemates problem, where there are n rooms rent by n housmates. Each housmate bids for every single room and finally pays a price for the room he or she gets (Su [66]). Allocating the rooms according to standard principles such as proportionality might lead to unattractive rents, e.g. paying more than one's bid, being paid to take a room, etc. Brams and Kilgour [18] developed a procedure which somehow avoids many problems arising with other allocation procedures.

[20] Extensive surveys have been written in this area, Moulin [47] [48] and Young [79] being just some of them.

[21] Other examples stem from medicine where a restricted amount of medicine needs to be divided among sick people with possibly different chances of survival. Also every tax system somehow has to solve the same problem, as the cost, i.e. the total tax necessary to run the state, needs to be raised from the taxpayers whose claims are their different income levels.

3.1.1 Proportional Method

If we consider again the starting example from the Talmud, we observe a clear recommendation of how to share the estate of e.g. $r = 100$ given the claims $x_1 = 100$, $x_2 = 200$ and $x_3 = 300$. This is an example of a deficit sharing problem as $r < \sum_{i \in N} x_i$. Let us check whether the Talmudian shares correspond to any of the intuitive suggestions on how to divide a resource. One first simple approach would be to divide the estate proportional to the agents' claims.

Definition 5. A rule ϕ is the **proportional rule** ϕ^p if and only if for all fair division problems (N, r, x), and all $i \in N$, $y_i = \phi_i(N, r, x) = \frac{x_i}{x_N} \cdot r$ for $x_N \equiv \sum_{i \in N} x_i > 0$.

Table 3 The proportional solution

estate	claims		
	100	200	300
100	$16\frac{2}{3}$	$33\frac{1}{3}$	50
200	$33\frac{1}{3}$	$66\frac{2}{3}$	100
300	50	100	150

The proportional rule treats agents according to their claims, by discounting each claim by the same factor. As we see from Table 3, the proportional method coincides with the Talmud only for $r = 300$.

3.1.2 Properties

The previous solution method seems reasonable, but is the division it suggests really "fair"? As in the previous section, fairness can be represented by different properties that such a solution might satisfy, only some of those used in the literature will be discussed in the following. It can easily be shown that the proportional method satisfies all of the following properties.

As a simple translation of envy-freeness from the previous framework is not possible, one of the most important properties, in the case that the claims vector is all the information that one is allowed to use, is that equals are treated equally.[22] This **equal treatment of equals property** says that if $x_i = x_j$ for some $i, j \in N$, then $y_i = y_j$. Another property, **efficiency**, requires in the usual form that all of the resource needs to be distributed, i.e. $\sum_{i \in N} y_i = r$, something easily satisfied by most methods. Equally interesting is the mild - but compelling - idea that any increase in the resource should not lead to any agent being worse off afterwards. This is what **resource monotonicity** guarantees[23], i.e. for all N, r, r' and x, if $r \leq r'$ then $\phi(N, r, x) \leq \phi(N, r', x)$. In a similar spirit, but with a focus on changes of claims, is the property **independence of merging and splitting** which says that for all $N, S \subseteq N$, all r and all x: $\phi(N, r, x)^{[S]} = \phi(N^{[S]}, r, x^{[S]})$. This implies that e.g. a merge of two different claims x_i, x_j to claim $x_{ij} = x_i + x_j$ should not change their joint share y_{ij}, i.e. $y_{ij} = y_i + y_j$. The same should hold if a claim is split into several parts. Finally, we might want to have a certain intuitive relationship between claims and shares, something that the property **fair ranking** requests, i.e. for all $i, j \in N$, $x_i \leq x_j \Rightarrow [y_i \leq y_j$ and $x_i - y_i \leq x_j - y_j]$.

3.1.3 Uniform Losses

Besides the idea of proportionality, one could hold other viewpoints. In the literature, two have been discussed widely (see Moulin [48]). One possibility is to distribute any surplus beyond x_N or deficit below x_N equally (in the deficit case with the restriction that nobody receives a negative share). Hence the claims do not have any real impact in how to

[22] The analog to this in cake-cutting would be that if $\mu_i(\cdot) = \mu_j(\cdot)$ for some $i, j \in N$, then the value of the pieces they receive should be identical.

[23] This is a sort of analog to **house monotonicity** in apportionment theory which is of interest w.r.t. the Alabama paradox. See Balinski and Young [4].

divide a surplus or deficit. This procedure is called the uniform losses rule (in the deficit case) or equal surplus rule (in the surplus case). The uniform losses rule is defined as follows:

Definition 6. A rule ϕ is the **uniform losses rule** ϕ^{ul} if and only if it associates the following solution for all $i \in N$ to any problem (N,r,x): $y_i = \phi_i(N,r,x) = (x_i - \delta)_+$, where $(x_i - \delta)_+ \equiv \max\{x_i - \delta, 0\}$ and δ is the solution of $\sum_{i\in N}(x_i - \delta)_+ = r$.

Table 4 The uniform losses solution

estate	claims		
	100	200	300
100	0	0	100
200	0	50	150
300	0	100	200
400	$33\frac{1}{3}$	$133\frac{1}{3}$	$233\frac{1}{3}$

Table 4 states the uniform losses solutions adding a fourth situation, namely $r = 400$. Compared to the Talmudian values, we see that the agent with the lowest claim has a disadvantage, as she gets 0 in the first three situations. This is due to the fact that the loss, i.e. the difference between the sum of the claims and the resource, is divided equally between the three agents (with the lower limit being zero).

3.1.4 Uniform Gains

Another - extremely egalitarian - option is to start from x and - in the deficit case - take away from those with the highest claims first as long as necessary, until all shares are equalized. Then reduce equally. In the surplus case start increasing the shares of those with the smallest claim as long as possible, until all shares are equalized. Then increase equally. It somehow alludes to the idea of a ”leximin”-ordering on the set of feasible solutions. This method is called the uniform gains rule[24] and defined as follows:

Definition 7. A rule ϕ is the **uniform gains rule** ϕ^{ug} if and only if it associates the following solution for all $i \in N$ to any problem (N,r,x): $y_i = \phi_i(N,r,x) = \min\{\lambda, x_i\}$ where λ is the solution of $\sum_{i\in N}\min\{\lambda, x_i\} = r$.

Table 5 The uniform gains solution

estate	claims		
	100	200	300
100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
200	$66\frac{2}{3}$	$66\frac{2}{3}$	$66\frac{2}{3}$
300	100	100	100
400	100	150	150

Table 5 presents the uniform gains solutions of our example. Its approach is extremely egalitarian and hence it favors the agent with the lowest claim compared to the previous methods. This is due to the fact that any deficit will first be covered by those with the highest claims.

A graphical representation of those rules is given in Figure 1 for $|N| = 2$. As can be seen for the deficit case[25], the uniform gains solution favors the agents with the smaller claims and the uniform losses solution those with the higher claims.

[24] It also has received other names in the literature such as Maimonides’ rule (Young [80]).

[25] In the surplus case, i.e. the resource line being beyond the claims point, the proportional rule would become most beneficial to the agents with higher claims.

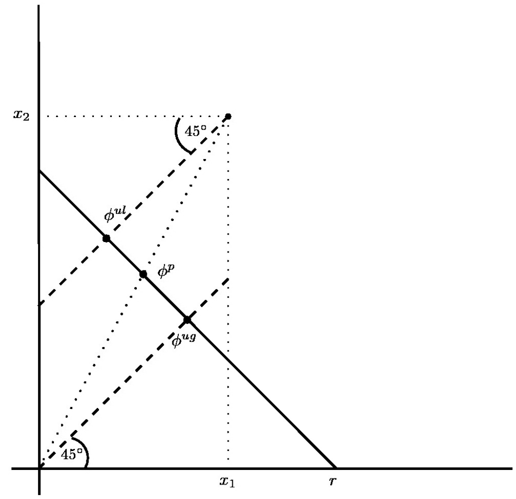

Fig. 1 Solutions for $n = 2$

How do the uniform gains and the uniform losses method fare w.r.t. the properties discussed above? Most of them are satisfied by ϕ^{ul} and ϕ^{ug} as well, however, consider the previous example and let agents 1 and 3 merge, i.e. we get a new situation in which $x_{13} = 400$ and $x_2 = 200$. If we assume $r = 300$, then we get the following solutions for the different methods presented in the first columns of Table 6.

Table 6 Merging and splitting

r=300	merging claims		splitting claims			
	400	200	100	200	150	150
ϕ^p	200	100	50	100	75	75
ϕ^{ul}	250	50	25	125	75	75
ϕ^{ug}	150	150	75	75	75	75

As can be seen in Table 6, the uniform losses method and the uniform gains method violate the property of indepence of merging and splitting. In the case of merging, $\phi^{ul}_{13} = 250$ and hence larger than the previous sum of their shares, $\phi^{ul}_1 + \phi^{ul}_3 = 200$. What happens in the case of splitting $x_3 = 300$ into two equally sized claims can be seen in the right columns of Table 6. Using uniform gains, the split increases the previous share of $\phi^{ug}_3 = 100$ to a joint share of 150. Only the proportional rule does not change the total shares of those that participate in merging or splitting.[26]

3.1.5 Contested Garment Method and Extensions

Given our historical example from the Talmud in Table 2, the three methods discussed so far do not formalize the recommendations made. Before tackling this problem in more detail we will discuss another - also historically very interesting - method dating back to the Talmud and extensively discussed in Aumann and Maschler [3]. The story goes

[26] The proportional, uniform losses and uniform gains methods are parametric methods. The first two of them belong to an important subclass of parametric methods, namely *equal sacrifice methods*. These are of relevance in taxation, where the x_i would represent taxable income and r the total aftertax income, making the difference $x_N - r$ the total tax raised. Given that, one can see that the three rules are important candidates for tax functions, with the proportional rule being both, progressive (average taxes do not decrease with income) and regressive (average taxes do not increase with income). Actually, the uniform gains method is the most progressive and the uniform losses method the most regressive among those rules satisfying fair ranking (see Moulin [48]).

as follows: Two men hold a garment, where one of them claims all of it and the other claims half. The Talmud suggests to give $\frac{3}{4}$ to the one who claimed it all and $\frac{1}{4}$ to the one who claimed half of it. This method - defined for two agents - is called the **Contested Garment Method** and is based on the idea of concessions, i.e. given the total resource, what would one agent concede to the other agent after having received all of his claim. E.g. consider only agents 1 and 2 from Table 2, i.e. $x_1 = 100$ and $x_2 = 200$, and let $r = 200$. This setting is equivalent to the story in the Talmud. Now, agent 1 only claims half of r and thus concedes 100 to agent 2, whereas agent 2 claims all of the resource and therefore concedes nothing to agent 1. The idea of the contested garment solution now is to allocate to the agents what they concede to each other and divide the rest equally. Formally, this can be written as follows:

Definition 8. For $|N| = 2$, a rule ϕ is the **contested garment method** ϕ^{cg} if for any problem (N,r,x) the shares are as follows:
$\phi_1 = \frac{1}{2}(r + \min\{x_1,r\} - \min\{x_2,r\})$
$\phi_2 = \frac{1}{2}(r - \min\{x_1,r\} + \min\{x_2,r\})$

In the example from the Talmud, agent 1 concedes 100 to agent 2, this leaves 100 to be divided equally and hence leads to shares $y = (50,150)$ as in the text. If we compare this to our previous three rules, we see that this solution is identical to the uniform losses solution. However, this is not always the case, as we can simply show by decreasing r to $r = 100$. Then $\phi^{cg} = (50,50)$, whereas $\phi^{ul} = (0,100)$.

There are two possibilities to generalize the contested garment method to $|N| \geq 2$. A first possibility is via the **Random-Priority method**. This works as follows: take a random order of N and let the agents take out their claims from the resource according to that order until there is nothing left. Do this for all possible orders of N and take the average.[27] For $|N| = 2$ this is identical to the contested garment solution. Again, let $r = 200$ and $x = (100,200)$, then if agent 2 goes first, he gets 200, if he goes second he gets 100. The average is exactly his contested garment share of 150. Doing the same for agent 1, we see that she receives 50 on average. We can also apply the random priority method to our previous Talmudic example. In this case, for $|N| = 3$, we get 6 rankings of the 3 agents. The shares are stated in Table 7 and are similar but still not identical to the numbers in the Talmud.

Table 7 The random priority method

estate	claims		
	100	200	300
100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
200	$33\frac{1}{3}$	$83\frac{1}{3}$	$83\frac{1}{3}$
300	50	100	150

Finally, we now get to the **Talmudic method** whose mathematical structure has been discovered by Aumann and Maschler [3] only in 1985, which provides the formalization of the divisions in Table 2. It is also another extension of the contested garment method, using an explicit mixture of the uniform gains and uniform losses methods.[28]

Definition 9. A sharing rule ϕ is the Talmudic method ϕ^t if and only if for all sharing problems (N,r,x), and all $i \in N$, $\phi_i = \phi_i^{ug}(N, \min\{r, \frac{1}{2}x_N\}, \frac{1}{2}x) + \phi_i^{ul}(N, (r - \frac{1}{2}x_N)_+, \frac{1}{2}x)$.

In words, the Talmudic method can be described as follows: Order the agents according to their claims in an increasing form, i.e. $x_1 \leq x_2 \leq ... \leq x_n$. Divide r equally until agent 1 has received a share of $\frac{x_1}{2}$ or all of r has been distributed. Eliminate agent 1 and continue increasing the shares of all other agents until agent 2 has received a share of $\frac{x_2}{2}$ or all of r has been used up. Eliminate agent 2 and continue the process as previously explained until either all agents have received a share of $\frac{x_i}{2}$ or the resource r has run out. If $r > \sum_{i \in N} \frac{x_i}{2}$, continue by increasing the share of agent n (the agent with the largest claim) until her loss, i.e. $x_n - y_n$, is equal to the loss of agent $n-1$ or the resource has been used up. Continue by increasing the shares of agents n and $n-1$ until their losses are equal to the loss of

[27] This has an obvious connection to the Shapley value.

[28] The Talmudic method and the Random Priority method are both self-dual, however the Talmudic method is the only consistent extension of the contested garment method. See Moulin [47].

agent $n-2$ or the resource has been eliminated. Repeat the process in the same way until all of the resource has been distributed.[29]

3.1.6 Indivisibilities

So far the value and/or the claims have been perfectly divisible. The situation slightly changes when claims, resource and shares need to be integers, e.g. because we are talking about the division of ”processing slots” of a number of jobs on a common server.[30] Those are so-called *queuing problems*, with x_i being the number of jobs requested by agent i and agents differing in the number of jobs they request. Those jobs need to be processed on a server, and the agents owning the jobs are impatient, i.e. agents want their jobs being processed as early as possible. The principal idea, however, is the same. Many of the previous properties remain unchanged. Symmetry is of course lost when we allocate indivisible goods, as long as the allocation is deterministic. Alternatively, we can also use probabilistic methods, and all of our proportional, uniform gains and uniform losses methods have probabilistic analogs. For a discussion of this part of the literature and various characterization results see Moulin [48] [49] [50] and Moulin and Stong [52].[31]

3.1.7 Incentives

If we are concerned with strategic aspects in this framework, the question arises, what happens if an agent's claim x_i is private information, so that she may be able to misrepresent her true claim? Obviously a rule such as the proportional method can be manipulated by increasing one's claim x_i, as $y_i = \frac{x_i}{x_N} \cdot r$ directly depends on x_i. Consider agent 3 mispresenting its claim by claiming $x_3' = 400$ instead of $x_3 = 300$. For $r = 300$ this gives shares as shown in Table 8:

Table 8 Manipulation possibilities

method	true claims			manipulated claim		
	100	200	300	100	200	400
ϕ^p	50	100	150	42.8	85.7	171.4
ϕ^{ul}	0	100	200	0	50	250
ϕ^{ug}	100	100	100	100	100	100

In our example, misrepresentation of agent 3 pays off for the proportional and the uniform losses method. The uniform gains method is not susceptible to manipulation in this setting. In all the literature on strategy-proofness, the uniform gains method stands out as the best method. Actually Sprumont [63] characterizes the uniform gains method by the properties strategy-proofness, efficiency and equal treatment of equals.[32]

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[29] An important aspect of those rules that relates this topic to cooperative game theory is the fact that both of them have well known couterparts in cooperative game theory. Aumann and Maschler [3] proved that the Talmudic method allocates the resources according to the nucleolus (of the appropriate games) and the Random-Priority method allocates the resources according to the Shapley value (of the appropriate games). Actually, it was via those counterparts that the Talmudic method has eventually been found. Thomson [75] evaluates certain of the above rules by looking at two families of rules. Among other things, he looks at duality aspects of the rules and offers characterisation results for consistent rules.

[30] This could be seen as the counterpart of the move from cake cutting to the division of indivisible items as in the previous section.

[31] Maniquet [45] provides a characterization of the Shapley value in queuing problems, combining classical fair division properties such as equal treatment of equals with properties specific to the scheduling model. He shows that the Shapley value solution stands out as a very equitable one among queuing problems.

[32] This is somehow based on the assumption of single-peaked preferences. See Thomson [74] for a discussion.

3.2 Division of variable resources/costs

A considerable change in the framework occurs whenever there is no fixed resource to be divided, but the resource is determined by individual demands. The typical example is sharing a joint cost created through the individual demands. As in the following we will focus on sharing costs, we define a cost-sharing problem as a triple (N,c,x) with N being the set of individuals, c being a continuous non-decreasing cost function $c:\Re_+ \to \Re_+$ with $c(0)=0$ and $x=(x_i)_{i\in N}$ specifying each agent's demand $x_i \geq 0$.

As before, a solution is a vector $y=(y_i)_{i\in N}$ specifying a cost share for each agent i s.t. $y_i \geq 0$ for all i and $\sum_{i\in N} y_i = c(\sum_{i\in N} x_i)$. A cost-sharing method ϕ is a mapping that associates to any cost-sharing problem a solution.

One cost sharing method which is an analog to the previously discussed proportional method when dividing a fixed resource, is the following:

Definition 10. The cost sharing method ϕ is the **average-cost method** ϕ^{ac} if and only if for all cost sharing problems (N,c,x), and all $i \in N$, $y_i = \phi_i^{ac}(N,c,x) = \frac{c(x_N)}{x_N} \cdot x_i$.

I.e. ϕ^{ac} divides total costs in proportion to individual demands. This method is informationally very economical, as it ignores any information about costs for just serving a certain subgroup. The closeness to the proportional method previously discussed is obvious. Moreover, if the properties, that characterize the proportional method are slightly modified, then a characterization result for the average cost method can be obtained (see Moulin [47]).

What happens if all of the information from the cost function c is used? Whenenver c is convex, i.e. demand becoming increasingly more costly, fairness seems to require that for any agent i, $\phi_i(N,c,x) \geq c(x_i)$, i.e. the amount agent i has to pay when the cost is shared between the whole group N is at least as large as if i demands x_i just for herself. In addition, another fairness consideration requires that the agent should not pay more than what she would have had to pay if all of the agents were like her, i.e. $\phi_i(N,c,x) \leq \frac{c(nx_i)}{n}$. For c being concave, the inequalities are reversed.[33]

For the following examples assume $|N|=3$, define $z=\sum_{i\in N} x_i$ and let the demand vector be $x=(1,2,3)$. Now, for a cost function $c(z)=\max\{0,z-4\}$, the shares of the cost $c(6)=2$, according to the average cost method, are $\phi^{ac}(N,c,x)=(\frac{1}{3},\frac{2}{3},1)$. However, agent 1 could claim that this is not fair, as if all others had been like him, then $c(3)=0$ and nobody would have had to pay anything. Another example would be the (increasing returns) cost function $c'(z) = \min\{\frac{z}{2}, 1+\frac{z}{6}\}$. The same vector $x=(1,2,3)$ now leads to the same shares as before, namely $\phi^{ac}(N,c',x)=(\frac{1}{3},\frac{2}{3},1)$. In this case however, agents 2 and 3 might challenge the low share of agent 1, as he is paying less than what he would have had to pay had all of the agents been like agent 1.

One way to tackle such problems is to take those differences into account. The following method has been suggested by Moulin and Shenker [51]:

Definition 11. Order the agents according to their demands, i.e. $x_1 \leq x_2 \leq \ldots \leq x_n$ and define $x^1 = nx_1$, $x^2 = x_1 + (n-1)x_2$, ..., $x^i = (n-i+1)x_i + \sum_{j=1}^{i-1} x_j$. The cost sharing method ϕ is the **serial cost-sharing method** ϕ^s if and only if for any cost sharing problem (N,c,x) the cost shares are $\phi_1(N,c,x) = y_1 = \frac{c(x^1)}{n}$, $\phi_2(N,c,x) = y_2 = y_1 + \frac{c(x^2)-c(x^1)}{n-1}$,...,$\phi_i(N,c,x) = y_i = y_{i-1} + \frac{c(x^i)-c(x^i-1)}{n-i+1}$.

Getting back to our previous numerical examples we get the shares $\phi^s(N,c,x) = (0,\frac{1}{2},\frac{3}{2})$ and $\phi^s(N,c',x) = (\frac{1}{2},\frac{2}{3},\frac{5}{6})$. In both cases the individual costs in relation to the cost function have been taken into account to determine the distribution.

A general fact is that the agent with the smallest demand prefers her serial cost share to her average cost share, when marginal costs are increasing, and vice versa with decreasing marginal costs.[34]

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[33] Depending on what the cost function looks like, this suggests upper and lower bounds on cost shares. For c being convex, the **stand-alone lower bound** $y_i \geq c(x_i)$ says that no agent can benefit from the presence of other users of the technology. In this sense we could think of other agents creating a *negative externality*. The opposite argumentation works for c being concave, creating a *positive externality*. Other bounds properties are discussed in the literature and used for characterization results. See Moulin [47].

[34] A further change in the framework would require the individual demands to be binary, i.e. $x_i \in \{0,1\}$. This moves us towards the model of cooperative games with transferable utility. The most famous method within this framework is the *Shapley value* (see Shapley [62]). See also Moulin [47] [48] for a discussion.

3.3 Fair Division on Graphs

The next change to our framework requires the introduction of a certain graph structure into our model. E.g. several towns, going to be connected to a common power plant, need to share the cost of the distribution network. There is a growing literature that analyses cost allocation rules in the case of a certain graph structure $G(N\cup\{0\},E)$, where $N\cup\{0\}$ denotes the set of nodes (i.e. set of agents plus the source $\{0\}$), and E is the set of edges, i.e. the set of all connections between any $i,j\in N\cup\{0\}$. In addition we have a cost function $c:E\rightarrow\Re_+$ that assigns to any edge $(ij)\in E$ a cost $c(ij)\geq 0$, denoting the cost of connecting node i with node j. Hence, a cost sharing problem in this framework is a pair (G,c).[35]

Efficient networks in such problems must be trees, which connect all agents to the source. Hence, a subset $T\subseteq E$ is called a spanning tree of G if the subgraph $(N\cup\{0\},T)$ of G is acyclic and connected. The set of all spanning trees is denoted by τ. Now, a spanning tree T is called minimum cost spanning tree if for all $T'\in\tau$, $\sum_{(ij)\in T'}c(ij)\geq\sum_{(ij)\in T}c(ij)$.[36] A cost sharing solution is now a vector of cost shares $y=(y_i)_{i\in N}\in\Re^n_+$ such that $\sum_{i\in N}y_i=c(T)$ where T is the minimum cost spanning tree of the problem (G,c).[37]

Consider the following example. Let $N=\{1,2,3\}$ form a network and connect to some common source $\{0\}$. The connection costs are as follows: $c(01)=4$, $c(02)=c(03)=5$, $c(12)=3$, $c(13)=6$ and $c(23)=2$. This can be represented by the following symmetric cost matrix M and Figure 2.

$$M=\begin{pmatrix}0&4&5&5\\4&0&3&6\\5&3&0&2\\5&6&2&0\end{pmatrix}$$

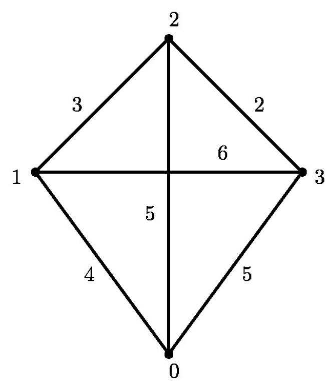

Fig. 2 Costs in a network

The task now is to connect all agents either directly or indirectly to the source and fairly divide the total cost among the agents based on the cost matrix. Bird [11] was one of the first to offer a solution to such cost allocation problems (G,c). First determine a minimal cost spanning tree. Now, starting from the source, each node (i.e. agent) has a predecessor in the spanning tree, namely the node that - on the path from the source to the agent along the spanning tree - comes immediately before that agent. Then the Bird method ϕ^B simply assigns to each agent the cost it takes to connect her with her predecessor.

Continuing the previous example, we see that the (unique) minimal cost spanning tree is $T=\{(01),(12),(23)\}$ with a total cost $c(T)=9$. Applying the Bird rule, we get the following cost shares for the agents: $\phi^B(G,c)=(4,3,2)$.

[35] If, instead of a cost structure, one uses preferences on the graph, a different framework arises in which the aggregate satisfaction of the agents determines the distribution network. Hence, this closely links this area with social choice theory. See e.g. Darmann et al. [25].

[36] In what follows we will slightly abuse the notation and define the cost of a spanning tree T as $c(T)\equiv\sum_{(ij)\in T}c(ij)$.

[37] The important thing is, that the structure of the problem implies that the domain of the allocation rule will be smaller than the domain in a more general cost sharing problem. This actually helps in creating allocation rules satisfying certain desirable properties, something that is impossible for larger domains (see e.g. Young [79]).

This solution is in the **core**, i.e. no coalition can block it by connecting independently to the source. Interestingly, the Bird rule always selects a solution in the core, which is of importance as Bird [11] shows that the core of a minimum cost spanning tree game is always non-empty.

However, it does have a serious drawback. Consider a slight change in the above cost matrix in the sense that $c'(03) = 3$ instead of 5. This changes the cost matrix M to M':

$$M' = \begin{pmatrix} 0 & 4 & 5 & 3 \\ 4 & 0 & 3 & 6 \\ 5 & 3 & 0 & 2 \\ 3 & 6 & 2 & 0 \end{pmatrix}$$

Moreover, it also changes the minimal cost spanning tree to $T' = \{(03), (23), (12)\}$ with a total cost of $c'(T') = 8$. Using Bird's rule we see that the new solution becomes $\phi^B(G, c') = (3, 2, 3)$, which goes against our intuition of fairness as agent 3's cost share in T' increased although the total cost in T' is lower than in T.

In general, this determines a major property in this literature, namely **cost monotonicity**. Whenever a cost matrix M changes to M' by decreasing just one entry $c(ij)$ in M, then neither i nor j should have a larger share in M'. This property can also be seen important in providing appropriate incentives to reduce costs.

Although the Bird rule violates cost monotonicity, Dutta and Kar [26] suggest a rule which is in the core and satisfies cost monotonicity. In contrast, Young [79] shows that in the context of transferable utility games, there is no solution concept which picks an allocation in the core of the game when the latter is nonempty and also satisfies a property which is analogous to cost monotonicity. However, in Dutta and Kar's [26] framework such a solution is possible because of the special structure of minimum cost spanning tree games. Monotonicity in that context is a weaker restriction. For a detailed discussion of their rule we refer to their original paper.

Finally, cost allocation rules coinciding with the Shapley value (which has been characterized by Kar [38]) satisfy cost-monotonicity but may lie outside the core.[38] This implies that some group of agents may well find it beneficial to construct their own network to reduce their cost shares.

4 Economics and Fair Division

The final approach to fair division, that we want to quickly discuss here, comes from the discipline of economics. It is rather of axiomatic nature and lies mostly within the classical economic models. An excellent survey has been provided by Thomson [74].

Again, the formal framework slightly differs from those used in the previous sections. Usually one starts with a set $N \equiv \{1, \ldots, n\}$ of agents and l commodities (privately appropriable and infinitely divisible). Each agent $i \in N$ is characterised by a preference relation R_i on consumption space $\Re^l_+$. The set of all preference relations on $\Re^l_+$ is denoted by $\mathscr{R}$. The vector of ressources available for distribution - the social endowment - is denoted by $\omega \in \Re^l_+$.

An economy is now just a pair (R, ω), namely a preference profile $R = (R_1, \ldots, R_n)$ and a social endowment ω.[39] Given an economy $e \equiv (R, \omega)$, a solution is an allocation $y = (y_1, \ldots, y_n) \in \Re^{ln}_+$, assigning to each agent i a commodity bundle $y_i \in \Re^l_+$. The set of all feasible solutions (allocations) for an economy e is denoted by $Z(e)$. The question is whether there are "fair" allocations (solutions) among $Z(e)$. As previously, "fairness" can be seen as the satisfaction of various properties by such allocations. Probably the most important single property is **no-envy** introduced by Foley [34] and defined as follows:[40]

Definition 12. An allocation y satisfies **no-envy** if $y_i R_i y_j$ for all $i, j \in N$.

I.e. each agent is at least as well off with her own bundle than with any other agent's bundle. Allocations satisfying no-envy always exist in this model as the equal division allocation obviously satisfies the definition of no-envy. Actually, given monotonic and convex preferences the existence of envy-free and Pareto efficient allocations can be

[38] For other (axiomatic) results in that respect see e.g. Bergantinos and Vidal-Puga [10] or Bogomolnaia and Moulin [12].

[39] Different models occur depending on the set $\mathscr{R}$, i.e. what the preferences look like (e.g. quasi-linear preferences) and the exact specification of ω.

[40] No-envy has the clear counterpart of envy-freeness used in cake-cutting. Other concepts related to no-envy - but not discussed here - do exist, such as average no envy, strict no-envy, balanced envy, etc. (see Thomson [74]).

shown simply by looking at Walrasian allocations. Envy-free and efficient allocations may not exist if preferences are not convex (Varian [76]). Some other widely used properties are the following:

- **No-domination**: A feasible allocation satisfies no-domination if for no $i, j \in N$ it is the case that $y_i \geq y_j$, i.e. no agent should get at least as much of all goods as another agent and strictly more of some good.[41]
- **Equal division lower bound**: An allocation y satisfies equal division lower bound if $yR\left(\frac{\omega}{n}, ..., \frac{\omega}{n}\right)$, i.e. each agent i considers her own bundle y_i at least as good as the bundle $\frac{\omega}{n}$.[42]
- **Equal treatment of equals**: An allocation y satisfies equal treatment of equals if for all $i, j \in N$, $R_i = R_j$ implies $y_i I_i y_j$ and $y_j I_j y_i$, i.e. both agents, i and j are indifferent between the bundles they receive.

For strictly monotonic preferences, no-envy implies no-domination. If convexity is not satisfied, then there are economies in which all efficient allocations violate no-domination (Maniquet [44]).

Other results show that in an efficient allocation, at least one agent envies nobody, and at least one agent is envied by nobody, whenever the feasible set is closed under permutations of the components of allocations (see Varian [76] and Feldman and Kirman [30]). It seems clear that otherwise this would lead to envy-cycles, which could be resolved by simply switching the bundles within the cycles, leading to a Pareto improvement w.r.t. the original allocation.

As - in exchange economies - there might exist many envy-free and efficient allocations, refinements have been looked at. One interesting approach is to provide a ranking of those allocations based on an index of fairness (Feldman and Kirman [30]).[43]

One important property is egalitarian equivalence introduced by Pazner and Schmeidler [58]. It involves comparisons to a reference allocation which - for certain, mostly obvious, reasons - is considered to be fair (but might be infeasible). More precisely, y is **egalitarian equivalent** for e, if there is a $y_0 \in \Re^l_+$ such that $yI(y_0, ..., y_0)$. It can be shown that in economies with strictly monotonic preferences, efficient and egalitarian equivalent allocations exist.

An aspect, that was also of relevance in the previous section, are changes in the endowment ω or the set of agents N. How should an allocation change when ω or N change? The following two monotonicity properties are based on such considerations:

- Let $e = (R, \omega)$ and $e' = (R, \omega')$ be two economies with $\omega \leq \omega'$. Then a method ϕ satisfies **resource monotonicity** if $y \in \phi(e)$ and $y' \in \phi(e')$ implies $y'Ry$.
- Let $e = (R_1, ..., R_{|N|}, \omega)$ and $e' = (R_1, ..., R_{|N'|}, \omega)$ be two economies with $N' \subset N$. Then a method ϕ satisfies **population monotonicity** if for all $i \in N'$, $\phi_i(e')R_i\phi_i(e)$.

Hence, resource monotonicity requires that any increase in the social endowment does not make anyone worse off. Population monotonicity demands that any decrease in the number of agents - given an unchanged social endowment - does not make any (previously already existing) agent worse off.

Moulin and Thomson [53] showed that for strictly monotonic, convex and homothetic preferences, any solution satisfying no-domination and efficiency violates resource monotonicity. Kim [41] showed for the same class of preferences that, given a sufficient number of potential agents, there is no population monotonic rule satisfying no-envy and efficiency.[44]

Tadenuma and Thomson [69] are concerned with the strategic aspects in such fair allocation situations and their extent of manipulability. They show that there is no non-manipulable subsolution of the no-envy solution. Moreover, they establish manipulation games and show that for any two solutions, not only are the sets of equilibrium allocations of their associated manipulation games identical, but also is this set of equilibrium allocations the same as the set of envy-free allocations for the true preferences.

Finally, Fleurbaey and Maniquet [32] use a similar model to analyse fair income tax schemes. They address the efficiency-equity trade-off - usually occuring because of distortions through the income tax - by constructing social preferences that take into account the standard Pareto principle as well as fairness conditions.

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[41] Observe that no preference information is used for this property.

[42] For any two vectors $y, y' \in \Re^{ln}_+$, we use yRy' for saying $y_iR_iy'_i$ for all $i \in N$.

[43] We can also create equity criteria for groups. This somehow is in the spirit of core properties from other areas. Many of the above properties can be translated into this framework, e.g. equal-division core of e, group envy-freeness, etc.

[44] All the results so far are based on private goods. Much less attention has been given to the study of fairness in the case of public goods, with the notable exception of e.g. Moulin [46] and Fleurbaey and Sprumont [33].

5 Conclusion

In this survey, we have tried to discuss some of the most important aspects of fair division. The goal was to emphasize how different disciplines such as mathematics, operations research or economics tackle such a problem. We saw that all fields started with slightly different frameworks depending on what it was that needed to be divided, what individual preferences looked like and whether the algorithmic or the axiomatic aspects were predominant. Changes in the framework can lead to new settings with new applications and possibly new interesting results and procedures.

It should have become apparent, that fairness can be defined in various different ways. Clearly, envy-freeness plays an important role in that respect, but we also saw that fairness could as well have something to do with monotonicity or efficiency or combinations of different properties. This also still opens the possibility for many new results on what fair division procedures could look like, i.e. what fairness could actually mean, and whether the joint satisfaction of certain properties might be feasible or not.

Moreover, it is not only the possible existence of a fair division, or the procedure that leads to a fair outcome, that is of importance. Fair division often involves the subjective announcement of preferences, something that usually is rather private information. This, however, attaches a certain relevance to strategic aspects. Devising procedures, which reduce the incentive for misrepresentation by the agents, is surely still an important research field.

To sum up, this survey should have given a short overview over different fair division procedures and the appropriate models to evaluate fairness aspects.

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