Claudia Sifel

Optically triggered Spin Entanglement of Electrons in Semiconductors

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Ao. Univ. Prof. Mag. Dr. Ulrich Hohenester
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Chapter 1

Introduction

Semiconductor quantum dots (QDs) are nanostructures in which carriers are confined in all three spatial directions on a length scale comparable to the de-Broglie wavelength [1, 2]. This results in a delta-like density of states reminiscent of natural atoms. For that reason QDs occupied by neutral multi-exciton complexes (i.e., multiple electron-hole pairs) are often referred to as *artificial atoms*. Occupancies with different numbers of electrons and holes result in charged exciton complexes which may be considered as *artificial ions*. When a QD is placed in a field-effect structure it becomes possible to charge the dot in a well controlled fashion. Because of the Coulomb blockade mechanism an external bias voltage allows to transfer electrons from a nearby n-type reservoir to the dot one after the other [3, 4].

These unique properties were exploited by Zrenner *et al.* [5] to fabricate a QD based device which allows to create a single-electron photocurrent through optical triggering. The schematic setup is depicted in Fig. 1.1: A short laser-pulse coherently excites an exciton in a quantum dot embedded in a field-effect structure; if the structure is properly designed, such that tunneling occurs on a much shorter timescale than radiative decay, the electron-hole excitation of the quantum dot decays into a separated electron and hole within the contacts, which is detected as the photocurrent. Within this scheme it thus becomes possible *to transfer optical excitations in a deterministic way to electrical currents.*

In this work we exploit this finding to propose a device which allows the optically triggered creation of a *spin-entangled electron pair*. The proposed structure (Fig. 1.2) is identical to the one used by Zrenner *et al.*, with the only exception that the dot is initially populated by a single surplus electron.

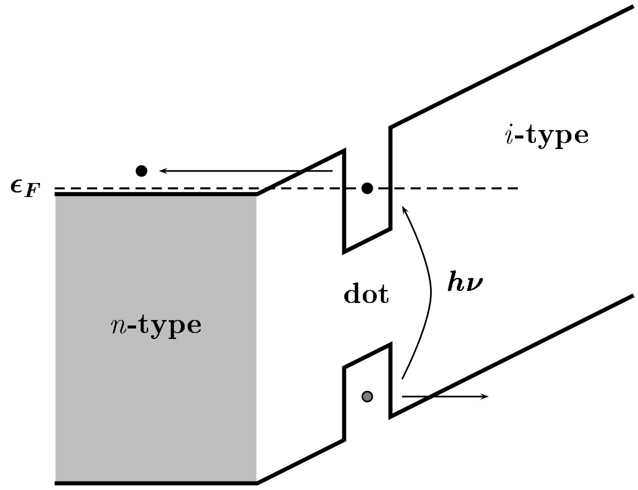

Figure 1.1: Band diagram of the structure used by Zrenner *et al.* [5]. For discussion see text.

This can be achieved by applying an external bias voltage such that an electron is transferred from a nearby *n*-type reservoir to the dot, where further charging is prohibited because of the Coulomb blockade. Optical excitation of this structure then results in the excitation of a charged exciton. Appropriate tuning of light polarization and frequency allows to selectively excite the charged-exciton groundstate, where the two electrons have opposite spin orientations. Since within the field-effect structure the charged exciton is not a stable configuration, in a consequent step one electron and hole will tunnel out from the dot to the nearby contacts; here, the system can follow two pathways, where either the electron in the dot has spin-up and the one in the reservoir spin-down orientation or vice versa. According to the laws of quantum mechanics, the total state of the system thus becomes a superposition of these two configurations; as will be proven in this work, *in this state the electron spins are maximally entangled.* Thus, the proposed device is an optically triggered turnstile for spin-entangled electrons, which could be used in future quantum information applications to establish entanglement between spatially separated sites.

In a sense, our scheme is similar to the proposal of Benson *et al.* [6] in which entangled photons are created in the cascade decay of a biexciton. However, in the system of our present concern additional difficulties arise

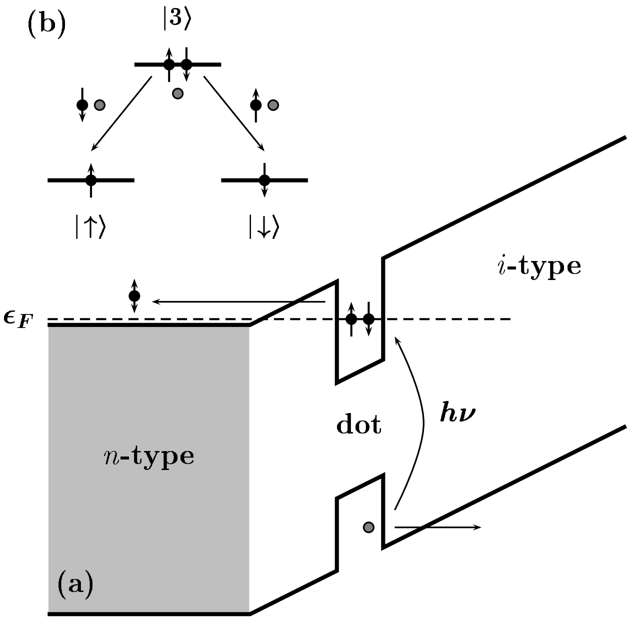

Figure 1.2: (a) Schematic band diagram of the proposed structure. (b) Level scheme of the spin-degenerate electron states $|\sigma = \uparrow, \downarrow\rangle$ and the charged-exciton state $|3\rangle$ in the dot.

because the tunnel-generated electron and hole do not propagate freely (as photons would in the corresponding scheme) but are subject to interactions in the contact. The resulting scatterings of the entangled particles hamper a straightforward interpretation of the functionality of the proposed device and call for a careful theoretical analysis. It is the purpose of this work to provide a comprehensible theory accounting for the complete cascade process of: the buildup of three-particle coherence through tunneling; the swapping of quantum coherence to spin entanglement through dephasing and relaxation in the reservoirs; and finally the process of disentanglement through spin-selective scatterings.

Within our theoretical approach we consider a QD interacting with its environment (i.e., *open system*) where the carriers are subject to different scattering channels. For low temperatures we can neglect phonon processes in the dot, and the tunneling process where one electron and hole tunnel out of the dot becomes the only relevant scattering. As for the electron in the reservoir, we assume that spin-unselective scattering processes take place on a much shorter time-scale (∼ps) than spin-selective scatterings (∼ns [7]). These different time-scales will allow us to separately describe these two processes, where the first ones lead to dephasing and relaxation, whereas the latter ones will destroy the spin entanglement. To account for such environment interactions in this work we use the framework of *density matrices* [8, 9], whose time evolution is computed through an unraveling of the master equation, i.e., *quantum-jump approach* [10–12]. Since the main emphasis of our work is on the identification of the basic schemes of the buildup and decay of entanglement, we rely on a simplified description scheme of these environment interactions, which will allow us to derive analytic expressions throughout.

This work has been organized as follows: In Chapter 2 the basic concepts of the density matrix approach, and of the solution of the master equation in Lindblad form through unraveling will be introduced. Next, in Chapter 3 our model of the interacting dot-reservoir system will be specified. In Chapter 4 we present the detailed analytic calculations. Finally, in Chapter 5 we give a short summary. For brevity, the technical details and the more lengthy calculations have been postponed to various appendices.

Chapter 2

Density-Operator Approach for Open Systems

2.1 Density Operator

A system interacting with its environment (*open system*) no longer can be described in terms of a pure wavefunction. Instead, a density-operator approach has to be adopted. It is the purpose of this chapter to introduce the basic properties of such an approach. I will start with the definition of the density operator and density matrix[1] for pure ensembles. Next, the definition of a density matrix for mixed ensembles will be introduced. Often it is the case that one is only interested in the properties of a part of the total system (i.e., subsystem), which can be described in terms of a *reduced density matrix.* At last, I will derive the equation of motion for the density matrix in the Schrödinger and interaction picture (Liouville-von-Neumann equation).

2.1.1 Density Operator for Pure Ensembles

A pure state $|\psi\rangle$ is characterized by the coefficients c_n of its expansion in a complete set of vectors $|u_n\rangle$ viz

$$|\psi\rangle = \sum_n c_n |u_n\rangle. \tag{2.1}$$

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[1] This is a possible mathematical representation of the density operator.

For a system in this state, an operator $\mathcal{A}$ has the expectation value

$$\begin{aligned}\langle\mathcal{A}\rangle &= \langle\psi|\mathcal{A}|\psi\rangle \\ &= \sum_{nn'} c_n c_{n'}^* \langle u_{n'}|\mathcal{A}|u_n\rangle. \end{aligned} \tag{2.2}$$

From the knowledge of $|\psi\rangle$ we can define the density matrix

$$\rho = |\psi\rangle\langle\psi| = \sum_{nn'} c_{n'}^* c_n |u_n\rangle\langle u_{n'}|, \tag{2.3}$$

which has the matrix elements

$$\rho_{nn'} = \langle n|\rho|n'\rangle = c_n c_{n'}^*. \tag{2.4}$$

The diagonal matrix elements ρ_{nn} are the probabilities to find the system in state $|u_n\rangle$, whereas the off-diagonal elements account for the quantum coherence. We emphasize that the density-matrix description remains valid when the system under investigation interacts with its environment, in contrast to the wavefunction description which is restricted to isolated systems.

With the help of the density operator expression (2.2) becomes

$$\langle\mathcal{A}\rangle = tr\{\rho\mathcal{A}\} = tr\{\mathcal{A}\rho\}, \tag{2.5}$$

where $tr\mathcal{B}$ is the trace, i.e., the sum over the diagonal elements of $\mathcal{B}$:

$$tr\ \mathcal{B} = \sum_n \langle n|\mathcal{B}|n\rangle. \tag{2.6}$$

Here $\{|n\rangle\}$ denotes a complete set of orthonormal states, i.e.[2],

$$\sum_n |n\rangle\langle n| = \mathbf{1}$$

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[2] Note that the trace is independent of the basis

$$\begin{aligned} tr\ \mathcal{B} = \sum_n \langle n|\mathcal{B}|n\rangle &= \sum_n \sum_m \langle n|m\rangle\langle m|\mathcal{B}|n\rangle \\ &= \sum_m \sum_n \langle m|\mathcal{B}|n\rangle\langle n|m\rangle = \sum_m \langle m|\mathcal{B}|m\rangle, \end{aligned}$$

for $\{|n\rangle\}$ and $\{|m\rangle\}$ being two different basis sets.

$$\langle l|n\rangle = \delta_{ln}. \tag{2.7}$$

Thus, Eq. (2.5) reads

$$tr\{\rho\mathcal{A}\} = \sum_n \langle n|\psi\rangle\langle\psi|\mathcal{A}|n\rangle = \sum_n \langle\psi|\mathcal{A}|n\rangle\langle n|\psi\rangle = \langle\mathcal{A}\rangle. \tag{2.8}$$

From the knowledge of the density operator we can calculate any expectation value of a system's observable. Quite generally, the restrictions on the matrix elements $\rho_{nn'}$ are [13]:

1. The condition that $\langle\mathcal{B}\rangle$ is real for every Hermitian operator $\mathcal{B}$ requires ρ to be Hermitian, i.e.,

   $$\rho = \rho^\dagger,$$

   $$\rho_{n'n} = \rho^*_{nn'}.$$

2. The condition that the unit operator $\mathbf{1}$ has the mean value 1 requires

   $$tr\ (\mathbf{1}\rho) = tr\ \rho = \sum_n \rho_{nn} = 1.$$

   Thus $tr\ \rho = \sum_n \rho_{nn}$ represents the total probability of finding the system in any state.

3. The condition that every operator with non-negative eigenvalues has a non-negative expectation value, requires ρ to be positive definite:

   $$\rho_{nn} \geq 0.$$

4. For pure ensembles ρ is idempotent, $\rho^2 = \rho$, but in general holds

   $$tr\ \rho^2 = \sum_{nn'} |\rho_{nn'}|^2 \leq 1.$$

2.1.2 Density Operator for Mixed Ensembles

When a system is in contact with its environment (open system), the density matrix can no longer be described by one pure state. Following the framework of statistical physics we introduce a description in terms of incoherent superpositions of pure states $|\psi_i\rangle$ with statistical weights p_i

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \overline{|\psi\rangle\langle\psi|}. \tag{2.9}$$

The density matrix elements are

$$\rho_{nn'} = \sum_i p_i \; c^{i*}_{n'} \; c^i_n = \overline{c^*_{n'} \; c_n}. \tag{2.10}$$

To each pure state corresponds an expectation value, $\langle \mathcal{A} \rangle_i = \langle \psi_i | \mathcal{A} | \psi_i \rangle$, which results for the mixed ensemble in

$$\langle \mathcal{A} \rangle = \sum_i p_i \langle \psi_i | \mathcal{A} | \psi_i \rangle = \sum_i p_i \langle \mathcal{A} \rangle_i = tr\{\rho \mathcal{A}\}, \tag{2.11}$$

where the last expression is completely analogous to that of the previous section.

The restrictions for the density matrix elements are the same as discussed before, but in general

$$\rho^2 = \sum_i \sum_j p_i p_j |\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j| \neq \rho$$

$$tr \; \rho^2 < 1, \text{if } p_i \neq 0 \text{ for more than one } i. \tag{2.12}$$

Therefore, the criterion for a pure state is $tr \; \rho^2 = 1$, and $tr \; \rho^2 < 1$ otherwise.

2.1.3 Reduced Density Operator

Assume that the total system is composed of two subsystems 1 and 2, with complete sets of orthonormal states $\{|n\rangle_1\}$ and $\{|m\rangle_2\}$, respectively. Then any pure state for the total system can be expanded as (see Appendix B)

$$|\psi\rangle = \sum_{n,m} c_{nm} |n\rangle_1 |m\rangle_2, \tag{2.13}$$

with

$$\sum_{n,m} |c_{nm}|^2 = 1. \tag{2.14}$$

The corresponding density matrix is

$$\rho = |\psi\rangle\langle\psi| = \sum_{n,m} \sum_{n',m'} c_{nm} c^*_{n'm'} |n\rangle_1 |m\rangle_2 \; {}_2\langle m'|_1\langle n'|. \tag{2.15}$$

In many cases we are only interested in the properties of system 1, whereas those of system 2 are irrelevant. In such cases, we can trace over system 2 viz

$$\rho_1 = tr_2\ \rho = \sum_{n,n',m} c_{nm} c^*_{n'm} |n\rangle_1\ {}_1\langle n'|. \tag{2.16}$$

ρ_1 is called the *reduced density matrix* of system 1.

Making a measurement on solely system 1 corresponds to an observable $\mathcal{A}$ which only acts on system 1. The expectation value of $\mathcal{A}$ is

$$\langle \mathcal{A} \rangle = tr_1\{\rho_1 \mathcal{A}\} = tr_1 tr_2\{\rho \mathcal{A}\}. \tag{2.17}$$

This expression shows that the knowledge of ρ_1 fully suffices to compute $\langle \mathcal{A} \rangle$.

In general holds $\rho_1^2 \neq \rho_1$. Only for a factorized state, i.e., $c_{nm} = a_n b_n$ with

$$\sum_n |a_n|^2 = 1, \ \sum_m |b_m|^2 = 1 \tag{2.18}$$

$\rho_1^2 = \rho_1$ holds. Otherwise ρ_1 represents the density matrix of a mixed ensemble.

Thus, even when the total system is in a pure state (2.13) ρ_1 might be represented by a mixed ensemble.

2.1.4 Liouville-Von-Neumann Equation

Liouville-Von-Neumann Equation in the Schrödinger Picture

In the last section we have seen that any quantum system can be described by a density operator ρ. Next, we derive its equation of motion.

To this end, we start from the time derivative of the density operator

$$\begin{aligned} \frac{\partial}{\partial t}\rho(t) &= \frac{\partial}{\partial t}\big(|\psi(t)\rangle\langle\psi(t)|\big) \\ &= |\dot{\psi}(t)\rangle\langle\psi(t)| + |\psi(t)\rangle\langle\dot{\psi}(t)|, \end{aligned} \tag{2.19}$$

where an analogous expression would hold for mixed ensembles. This can be cast with the Schrödinger equation ($H = H^\dagger$ is a Hermitian Hamiltonian)

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle \tag{2.20}$$

$$-i\hbar\frac{\partial}{\partial t}\langle\psi(t)| = \langle\psi(t)|H \tag{2.21}$$

to the final form

$$\frac{\partial}{\partial t}\rho(t) = -\frac{i}{\hbar}[H, \rho(t)]. \tag{2.22}$$

This is the *Liouville-von-Neumann equation* (also referred to as the von-Neumann equation), which is the quantum mechanical analog of the Liouville-equation in classical statistical mechanics.

The density operator evolves in time according to the unitary time-evolution operator U, i.e., $U^\dagger U = UU^\dagger = 1$. This can be seen from the formal solution of the Schrödinger equation

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}H(t-t_0)}|\psi(t_0)\rangle = U(t, t_0)|\psi(t_0)\rangle, \tag{2.23}$$

which results in

$$\rho(t) = U(t, t_0)\rho(t_0)U^\dagger(t, t_0), \tag{2.24}$$

where $\rho(t_0) = |\psi(t_0)\rangle\langle\psi(t_0)|$ is the initial density operator.

We now focus on the case that the total system can be separated into two parts S and R, where S is the system we are interested in, and R is the reservoir which will be traced over. Correspondingly, we divide the Hamiltonian into the parts

$$H = H_S + H_R + H_1, \tag{2.25}$$

where H_S and H_R are the Hamiltonians for S and R, respectively, and H_1 is the interaction between them. The reduced density operators of S and R, $\rho_S(t)$ and $\rho_R(t)$, can be extracted from $\rho(t)$ by tracing over the undesired parts of the Hilbert space viz

$$\rho_S(t) = tr_R\ \rho(t)$$

$$\rho_R(t) = tr_S\ \rho(t). \tag{2.26}$$

They obey the *master equations*

$$\dot{\rho}_S(t) = -\frac{i}{\hbar}\ tr_R\ \{[H, \rho(t)]\}$$

$$\dot{\rho}_R(t) = -\frac{i}{\hbar}\ tr_S\ \{[H, \rho(t)]\}. \tag{2.27}$$

Liouville-Von-Neumann Equation in the Interaction Picture

It is often convenient to change from the Schrödinger to the interaction picture. We first rewrite the Hamiltonian (2.25) as

$$H = H_0 + H_1, \tag{2.28}$$

where $H_0 = H_S + H_R$.

The density operator in the interaction picture $\rho^I(t)$ is obtained from $\rho(t)$ via the unitary transformation

$$\rho^I(t) = e^{\frac{i}{\hbar}H_0 t}\ \rho(t)\ e^{-\frac{i}{\hbar}H_0 t}, \tag{2.29}$$

as discussed in more detail in Appendix A. Differentiating this equation with respect to time

$$\dot{\rho}^I(t) = e^{\frac{i}{\hbar}H_0 t}\Big\{\frac{i}{\hbar}[H_0, \rho(t)] + \dot{\rho}(t)\Big\}e^{-\frac{i}{\hbar}H_0 t}, \tag{2.30}$$

and making use of the relation between operators in the Schrödinger and interaction picture

$$\rho(t) = e^{-\frac{i}{\hbar}H_0 t}\ \rho^I(t)\ e^{\frac{i}{\hbar}H_0 t}, \tag{2.31}$$

we obtain the equation of motion

$$\dot{\rho}^I(t) = -\frac{i}{\hbar}[H_1^I(t), \rho^I(t)]. \tag{2.32}$$

In comparison to the Schrödinger picture the density operator in the interaction picture evolves only due to the interaction-Hamiltonian $H_1^I(t)$.

Lowest Order Perturbation Theory

Next, we show how to formally solve Eq. (2.32) in lowest order perturbation theory. We formally integrate the Liouville-von-Neumann equation (2.32), and obtain

$$\rho^I(t) = \rho^I(t_0) - \frac{i}{\hbar}\int_{t_0}^{t} dt'[H_1^I(t'), \rho^I(t')]. \tag{2.33}$$

Substituting back to (2.32) gives

$$\dot{\rho}^I(t) = \frac{1}{i\hbar}[H_1^I(t), \rho^I(t_0)] - \frac{1}{\hbar^2}\int_{t_0}^{t} dt'[H_1^I(t), [H_1^I(t'), \rho^I(t')]. \tag{2.34}$$

If the system is composed of two parts S and R, and we are interested only in system S, we can trace over the variables of system R to get for the reduced density operator of the system S

$$\dot{\rho}_S^I(t) = -\frac{1}{\hbar^2}\int_{t_0}^{t} dt' \; tr_R \; \{[H_1^I(t), [H_1^I(t'), \rho^I(t')]\}. \tag{2.35}$$

In the last step we have assumed that

$$tr_R \; \{[H_1^I(t), \rho^I(t_0)]\} = 0, \tag{2.36}$$

which is justified when at time $t = t_0$ there is no correlation between systems S and R (i.e., $\rho^I(t_0) = \rho_S^I(t_0) \otimes \rho_R^I(t_0)$).

To evaluate the double commutator in (2.35) we make some assumptions. Let us expand $\rho^I(t')$ under the integral in (2.35) in a Taylor series around $t' = t$,

$$\rho^I(t') = \rho^I(t) + (t' - t)\dot{\rho}^I(t) + \ldots, \tag{2.37}$$

and assume that $\rho^I(t')$ varies sufficiently slow in the time interval of integration, we can replace $\rho^I(t')$ by its most recent value $\rho^I(t)$

$$\dot{\rho}_S^I(t) = -\frac{1}{\hbar^2}\int_{t_0}^{t} dt' \; tr_R \; \{[H_1^I(t), [H_1^I(t'), \rho^I(t)]\}. \tag{2.38}$$

For this reason the procedure is sometimes known as the short memory or *Markov approximation* [3] [8, 9] .

Our strategy to derive a master equation for systems coupled to reservoirs thus is: start with the Liouville-von-Neumann equation to lowest order in perturbation theory, trace over the reservoir, perform the Markov and *adiabatic approximations* (i.e., let t_0 in equation (2.35) approach negative infinity, $\lim t_0 \to -\infty$), and switch then to the Schrödinger picture.

2.2 Lindblad Operator

In section 2.1.4 we derived the master equation for a reduced density operator. In general, the master equation can be written as

$$\dot{\rho} = -\frac{i}{\hbar}[H_S, \rho] + \mathcal{L}[\rho], \tag{2.39}$$

where ρ describes system S, H_S is the Hamiltonian acting on S, and $\mathcal{L}[\rho]$ describes all environment couplings [14, 15]. A special form of $\mathcal{L}[\rho]$ is the *Lindblad form*[4]

$$\mathcal{L}[\rho] = -\frac{1}{2}\sum_i \left(L_i^\dagger L_i \rho + \rho L_i^\dagger L_i\right) + \sum_i L_i \rho L_i^\dagger, \tag{2.40}$$

where the L_i's are system operators, called Linblad operators. The summation goes over different scattering channels for the system under investigation.

The Liouvillian $\mathcal{L}[\rho]$ describes the non-unitary evolution of the system due to its coupling to the reservoir, and is responsible for decoherence and dissipation.
With Eq. (2.40) we obtain for the master equation (2.39)

$$\dot{\rho} = -\frac{i}{\hbar}\left(H_{\text{eff}}\rho - \rho H_{\text{eff}}^\dagger\right) + \mathcal{L}_{\text{jump}}[\rho], \tag{2.41}$$

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[3] In lowest order perturbation theory this is a reasonable assumption since an interaction of systems with reservoirs leads to damping which quickly destroys the memory of the past.

[4] Open quantum systems in the Markov limit (yielding to a master equation) are commonly described by density operators which evolve according to a master equation in Lindblad form.

with the non-Hermitian effective Hamiltonian

$$H_{\text{eff}} = H_S - \frac{i\hbar}{2}\sum_i L_i^\dagger L_i \tag{2.42}$$

and the 'jump' Liouvillian

$$\mathcal{L}_{\text{jump}}[\rho] = \sum_i L_i \rho L_i^\dagger . \tag{2.43}$$

We will see below that the evolution of the reduced density operator can be split into a Schrödinger-like part governed by the effective Hamiltonian H_{eff}, and a *'quantum jump'* part resulting from $\mathcal{L}_{\text{jump}}[\rho]$.

2.3 Unraveling of the Master Equation

In recent years a number of techniques has been developed for the solution of the master equation in Lindblad form (*unravelings of the master equation*). The most common are:

- The quantum-jump approach or Monte Carlo wavefunction method [10–12].
- The quantum-state diffusion (QSD) approach [16].

All this methods base on the wavefunction ('*quantum trajectories*' [17]) to describe a system coupled to the reservoir. I will treat the first one because it is more simple, and well suited for the solution of the problem studied in this work. Remind that the density operator is a statistical mixture of state vectors,

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \tag{2.44}$$

where the summation over i results from a statistical average over the various pure states $|\psi_i\rangle$. As a next step, we insert (2.44) into the master equation (2.41) to obtain for each contribution $|\psi_i\rangle\langle\psi_i|$

$$\begin{aligned} |\dot{\psi}_i\rangle\langle\psi_i| + |\psi_i\rangle\langle\dot{\psi}_i| &= -\frac{i}{\hbar}\left(H_{\text{eff}}|\psi_i\rangle\langle\psi_i| - |\psi_i\rangle\langle\psi_i|H_{\text{eff}}^\dagger\right) \\ &\quad + \sum_j L_j|\psi_i\rangle\langle\psi_i|L_j^\dagger . \end{aligned} \tag{2.45}$$

Restricting ourselves to a single state vector $|\psi\rangle$ in the mixture, we recognize that the first term on the right-hand side can be interpreted as a non-Hermitian (but Schrödinger-like) evolution of $|\psi\rangle$ under the influence of H_{eff},

$$i\hbar|\dot{\psi}\rangle = H_{\text{eff}}|\psi\rangle. \tag{2.46}$$

In contrast, the second term represents a discontinuous time evolution where the state $|\psi\rangle$ is projected — or 'jumps' — onto one of the possible states

$$|\psi\rangle \rightarrow |\psi\rangle_j = L_j|\psi\rangle. \tag{2.47}$$

This is the motivation for calling $\mathcal{L}_{\text{jump}}[\rho]$ a 'quantum jump' Liouvillian.

The decomposition of the evolution of the state vector $|\psi\rangle$ into a Schrödinger-like part and a quantum-jump part suggests an elegant way to solve the master equation.

We will now show how to calculate the change of the wavefunction

$$|\psi(t)\rangle \rightarrow |\psi(t+\delta t)\rangle$$

due to the Schrödinger-like and quantum-jump part, respectively. For sufficiently small time intervals, the time evolution according to H_{eff} is given by

$$|\tilde{\psi}(t+\delta t)\rangle = \big(1 - \frac{i}{\hbar}H_{\text{eff}}\delta t\big)|\psi(t)\rangle. \tag{2.48}$$

Note that H_{eff} is non-Hermitian, and consequently $|\tilde{\psi}(t+\delta t)\rangle$ is not normalized. The square of its norm[5] is

$$\begin{aligned}
&\|\ \langle\tilde{\psi}(t+\delta t)|\tilde{\psi}(t+\delta t)\rangle\|^2 = \langle\tilde{\psi}(t+\delta t)|\tilde{\psi}(t+\delta t)\rangle \\
&= \langle\psi(t)|\big(1 + \frac{i}{\hbar}H_{\text{eff}}^{\dagger}\delta t\big)\big(1 - \frac{i}{\hbar}H_{\text{eff}}\delta t\big)|\psi(t)\rangle \\
&= \langle\psi(t)|\big(1 + \frac{i}{\hbar}(H_{\text{eff}}^{\dagger}\delta t - H_{\text{eff}}\delta t) + \mathcal{O}(\delta t^2)\big)|\psi(t)\rangle \\
&= 1 - \delta p,
\end{aligned} \tag{2.49}$$

---

[5] Which is the probability to be in the state $|\tilde{\psi}(t+\delta t)\rangle$.

where to lowest order in δt

$$\begin{aligned}\delta p &= \frac{i}{\hbar}\delta t\langle\psi(t)|H_{\text{eff}} - H_{\text{eff}}^{\dagger}|\psi(t)\rangle \\ &= \delta t\sum_j\langle\psi(t)|L_j^{\dagger}L_j|\psi(t)\rangle \equiv \sum_j \delta p_j. \qquad (2.50)\end{aligned}$$

The full master equation evolution has to preserve the norm. Apparently, the lack of norm conservation results from the fact that so far we have ignored the effects of $\mathcal{L}_{\text{jump}}[\rho]$.

The 'missing norm' δp must therefore be brought in by the states $|\psi\rangle_j$, which result from the jump parts of the evolution. $\mathcal{L}_{\text{jump}}[\rho]$ projects the system onto the state $|\psi\rangle_j = L_j|\psi\rangle$, with probabilities δp_j, and $\sum_i \delta p_j = \delta p$.

Recapitulating [14, 15],

- if no jump occurs then

$$|\psi(t+\delta t)\rangle = \frac{\left(1 - \frac{i}{\hbar}H_{\text{eff}}\delta t\right)|\psi(t)\rangle}{\left\|\left(1 - \frac{i}{\hbar}H_{\text{eff}}\delta t\right)|\psi(t)\rangle\right\|}, \qquad (2.51)$$

- if a jump occurs, according to the relative probability among the various possible types of jumps, $P_j = \delta p_j/\delta p$ with $\sum_j P_j = 1$, then

$$|\psi(t+\delta t)\rangle = \frac{L_j|\psi(t)\rangle}{\|L_j|\psi(t)\rangle\|} = \frac{L_j|\psi(t)\rangle}{\sqrt{P_j}}. \qquad (2.52)$$

2.3.1 Decomposition of the Density Operator

According to this discussion, the time evolution of the density operator $\rho = |\psi\rangle\langle\psi|$ can be decomposed into two parts

$$\rho(t) \to \rho(t+\Delta t) = (1-\Delta P)|\psi_0\rangle\langle\psi_0| + \Delta P|\psi_1\rangle\langle\psi_1|, \qquad (2.53)$$

i.e., a part of no scattering with probability $(1-\Delta P)$ and a part of scattered states ('jumps') with probability ΔP. For that reason, the different parts of ρ in Eq. (2.53) are often referred to as *conditional density operators*.

To proof this decomposition, we express the normalized state vector for a jump, according to Eq. (2.52), as

$$|\psi_1\rangle = \frac{L}{\sqrt{\Delta P}}|\psi\rangle, \tag{2.54}$$

and for no jumps, according to Eq. (2.51)[6], the state evolves with

$$|\psi_0\rangle = \frac{(1 - i\ H_{\text{eff}}\ \Delta t)}{\sqrt{(1-\Delta P)}}\ |\psi\rangle, \tag{2.55}$$

This leads to

$$\begin{aligned} \rho &= \Delta P|\psi_1\rangle\langle\psi_1| + (1-\Delta P)|\psi_0\rangle\langle\psi_0| \\ &= \Delta t\ L\ |\psi\rangle\langle\psi|\ L^\dagger + (1 - i\ H_{\text{eff}}\ \Delta t)\ |\psi\rangle\langle\psi|\ (1 + i\ H_{eff}\ \Delta t) \\ &\simeq |\psi\rangle\langle\psi| - i\ \Delta t\ [H_S, |\psi\rangle\langle\psi|] - \frac{1}{2}\Delta t\ [L^\dagger L, |\psi\rangle\langle\psi|]_+ \\ &+ \Delta t\ L\ |\psi\rangle\langle\psi|\ L^\dagger + \mathcal{O}((\Delta t)^2). \end{aligned} \tag{2.56}$$

Neglecting terms of order $(\Delta t)^2$, and taking the 'time-derivative' of the density operator

$$\frac{\Delta\rho}{\Delta t} \simeq -i[H_S, \rho] - \frac{1}{2}[L^\dagger L, |\psi\rangle\langle\psi|]_+ + L|\psi\rangle\langle\psi|L^\dagger \tag{2.57}$$

we recover the initial master equation (2.39), which finally proofs our decomposition (2.53).

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[6] Henceforth we set $\hbar = 1$.

Chapter 3

System-Reservoir Model

In this chapter we specify our model system introduced in Chapter 1, which consists of a quantum dot (QD) coupled to nearby reservoirs (treated as Fermi seas of noninteracting electrons and holes). Neglecting at first environment interactions, this system is described by the total Hamiltonian

$$\begin{aligned} H &= H_0^D + H_R + V_T \\ &= H_0 + V_T, \end{aligned} \tag{3.1}$$

where $H_0 = H_0^D + H_R$ denotes the Hamiltonian of the QD and of the reservoirs, whereas V_T describes the tunnel process from the QD into the reservoirs.

For low temperatures and appropriate optical excitation, the unperturbed Hamiltonian of the dot system can be restricted to (see Fig. 1.2 b)

$$H_0^D = \sum_\sigma \epsilon_\sigma |\sigma\rangle\langle\sigma| + \epsilon_3 |3\rangle\langle 3|. \tag{3.2}$$

Here, the sum goes over the possible spin orientations $\sigma = \uparrow, \downarrow$ of the electron in the dot; $|\sigma\rangle$ denotes the spin-degenerate electron ground states with energy ϵ_σ; and $|3\rangle$ is the ground state [1] of the negatively charged exciton, with energy ϵ_3. The charged exciton X^- is a Coulomb renormalized complex consisting of two electrons and a hole.

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[1] The dot basis fulfills the completeness relation and is orthonormal: $\sum_\sigma |\sigma\rangle\langle\sigma| + |3\rangle\langle 3| = 1$, $\langle\sigma|\sigma'\rangle = \delta_{\sigma\sigma'}$, $\langle\sigma|3\rangle = 0$.

The unperturbed Hamiltonian of the reservoirs R consists of a continuum of quantum states for electrons and holes (which are orthogonal to the dot states). We describe the corresponding states in terms of electron and hole creation operators $c_{k\sigma}^\dagger$ and $d_k^\dagger$, respectively (and annihilation operators $c_{k\sigma}$ and d_k). Within this framework the reservoir Hamiltonian is of the form

$$H_R = \sum_{k\sigma} \epsilon_{k\sigma}^e c_{k\sigma}^\dagger c_{k\sigma} + \sum_k \epsilon_k^h d_k^\dagger d_k, \tag{3.3}$$

where the sum runs over the spin orientation σ and quantum numbers k (e.g. wavevector and band index). Note that for simplicity we do not explicitely indicate the hole spin. Since we assume fast hole relaxation, below we will trace out the hole degrees of freedom in the reservoir and solely focus on the entanglement of the two electrons after tunneling and scatterings in the reservoir.

The model for our system-reservoir interaction, the tunnel-Hamiltonian, can be written as

$$V_T = -\sum_{kk',\sigma} \left[t_{kk',\sigma}\; c_{k\sigma}^\dagger\; d_{k'}^\dagger\; |\bar{\sigma}\rangle\langle 3| + h.c. \right], \tag{3.4}$$

where $h.c.$ stands for *Hermitian conjugate*, and $\bar{\sigma}$ is the opposite (antiparallel) spin orientation of σ. The first term on the right-hand side of (3.4) reads from right to left as follows: The tunnel process destroys the charged exciton and creates instead an electron in the QD and a hole plus an electron in the reservoirs. The matrix elements $t_{kk',\sigma}$ are the strengths for tunneling out of the dot. The various spin-unselective and spin-selective scatterings will be treated within the framework of Lindblad operators, as will be described in more detail in the following chapter.

The main emphasis of our work is on the identification of the basic schemes underlying the buildup and decay of entanglement through environment interactions. Here, the main questions to be answered are:

- How are the quantum properties of the charged exciton transferred to the separated electrons and hole after tunneling?
- How is spin entanglement affected by spin-unselective scatterings?
- Do spin-selective scatterings in the reservoir act as quantum measurements, which allow to control the spin state of the dot electron?

Chapter 4

Tunneling, Decoherence, and Disentanglement

4.1 The Tunnel Process

4.1.1 Derivation of the Master Equation

First, we introduce some further simplifications to bring the tunneling Hamiltonian into a more convenient form. We assume that the matrix elements $t_{kk',\sigma}$ depend only on the electron and hole energies viz

$$t_{kk',\sigma} = \hat{t}(\epsilon^{e}_{k\sigma}, \epsilon^{h}_{k'}). \tag{4.1}$$

In addition, we introduce the field operators for electrons and holes

$$C_\sigma(\omega) = \sum_k c_{k\sigma}\delta(\omega - \epsilon^{e}_{k\sigma}) \tag{4.2}$$

$$D(\omega) = \sum_k d_k\delta(\omega - \epsilon^{h}_{k}), \tag{4.3}$$

respectively. These operators depend only on the electron and hole energies. They will allow us to answer the questions of the last chapter for a prototypical (although simplified) description of the various scattering channels.

Because electrons and holes are fermions they obey the usual anticommutator relations

$$[C_\sigma(\omega), C_{\sigma'}(\omega')]_+ = [C^\dagger_\sigma(\omega), C^\dagger_{\sigma'}(\omega')]_+ = 0$$

$$[C_\sigma(\omega), C^\dagger_{\sigma'}(\omega')]_+ = \delta_{\sigma\sigma'}\delta(\omega-\omega')g^e_\sigma(\omega) \tag{4.4}$$

$$[D(\omega), D(\omega')]_+ = [D^\dagger(\omega), D^\dagger(\omega')]_+ = 0$$

$$[D(\omega), D^\dagger(\omega')]_+ = \delta(\omega-\omega')g^h(\omega). \tag{4.5}$$

Here, we have introduced the density of states for electrons and holes

$$g^e_\sigma(\omega) = \sum_k \delta(\omega - \epsilon^e_{\sigma k})$$

$$g^h(\omega) = \sum_k \delta(\omega - \epsilon^h_k), \tag{4.6}$$

respectively, where $\delta(\omega)$ is Dirac's δ-function.[1]

Note that the electron field operators C and $C^\dagger$ anticommute with the hole operators D and $D^\dagger$.

Using the above quantities the tunneling Hamiltonian (3.4) becomes

$$V_T = -\sum_\sigma \int d\omega_e d\omega_h \big[\hat{t}(\omega_e,\omega_h)\; C^\dagger_\sigma(\omega_e)\; D^\dagger(\omega_h)\; |\bar{\sigma}\rangle\langle 3| + h.c.\big]. \tag{4.7}$$

To derive a master equation we proceed as follows. First, we suppose that at the initial time t_0 the dot system is described by a density operator $\rho_D(t_0)$. We assume that initially there is no correlation between dot and reservoir

$$\rho(t_0) = \rho_D(t_0) \otimes \rho_R(t_0). \tag{4.8}$$

This is justified if we assume that at early times the dot is populated by a single surplus electron, and at time t_0 a short laser pulse excites the charged exciton (such that no tunneling takes place during optical excitation).

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[1] Equation (4.6) makes only sense if integrated over it, $\int_{E_1}^{E_2} d\omega\delta(\omega-\epsilon_k)$, because of the δ-functions. If the energy ϵ_k of a state lies within the range of integration the integral over $\delta(\omega-\epsilon_k)$ gives unity. If the energy ϵ_k lies outside the range of integration, on the other hand, there is no contribution because the weight of the δ-function is concentrated entirely at $\omega=\epsilon_k$. Thus the integral gives 1 for all states in the range $E_1 \leq \epsilon_k \leq E_2$ and zero for those outside. Performing the sum, we see that it adds up to the total number of states between E_1 and E_2. Therefore the name 'density of states'.

To go further in our analytical calculations, we consider the Liouville-von-Neumann-equation for the total density operator of the system in lowest order perturbation theory and within the interaction picture, as discussed in Sect. 2.1.4,

$$\dot{\rho}^I(t) = -\int_{t_0}^{t} d\bar{t}\ [V_T^I(t), [V_T^I(\bar{t}), \rho^I(t)]]. \tag{4.9}$$

To evaluate the integral (4.9) we consider the tunnel-Hamiltonian in the interaction picture[2]

$$\begin{aligned} V_T^I(\bar{t}) \;\; &= \;\; -\sum_{\sigma} \int d\omega_e d\omega_h \big[\hat{t}(\omega_e, \omega_h)\ e^{i(\omega_e+\omega_h)\bar{t}} \\ &\qquad C_\sigma^\dagger(\omega_e)\ D^\dagger(\omega_h)\ e^{i(\epsilon_{\bar{\sigma}}-\epsilon_3)\bar{t}}\ |\bar{\sigma}\rangle\langle 3| + h.c.\big], \end{aligned} \tag{4.10}$$

and use the adiabatic approximation; here, we let the initial time t_0 approach negative infinity,

$$\lim_{t_0\to-\infty} \int_{t_0}^{t} d\bar{t}\ e^{\pm i\Omega\bar{t}} \simeq \pi\ \delta(\Omega). \tag{4.11}$$

Note that we have neglected Cauchy's principal part, which is a reasonable assumption since we are only interested in the scattering dynamics and not in energy-renormalization effects [18].

This yields to a new expression without explicit time-dependence

$$\begin{aligned} \lim_{t_0\to-\infty} \;\; \textstyle\int_{t_0}^{t} \;\; & d\bar{t}\ V_T^I(\bar{t}) = -\pi\ \delta(\omega_e + \omega_h + \epsilon_{\bar{\sigma}} - \epsilon_3)\ \times \\ \textstyle\sum_{\sigma} \;\; & \int d\omega_e d\omega_h \big[\hat{t}(\omega_e, \omega_h)\ C_\sigma^\dagger(\omega_e)\ D^\dagger(\omega_h)\ |\bar{\sigma}\rangle\langle 3| \\ + \;\; & \hat{t}^*(\omega_e, \omega_h)\ D(\omega_h)\ C_\sigma(\omega_e)\ |3\rangle\langle\bar{\sigma}|\big] \\ = \;\; & A_T. \end{aligned} \tag{4.12}$$

For convenience, we next transform back to the Schrödinger picture. This task is accomplished by the following procedure. Differentiating

$$\rho^I(t) = e^{iH_0 t}\ \rho(t)\ e^{-iH_0 t} \tag{4.13}$$

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[2] Note that we set $\hbar = 1$ in all our calculations.

with respect to time we get

$$\dot{\rho}^I(t) = e^{iH_0 t}\ (\dot{\rho}(t) + i[H_0, \rho(t)])\ e^{-iH_0 t}. \tag{4.14}$$

Thus

$$-i[H_0, \rho(t)] + e^{-iH_0 t}\ \dot{\rho}^I(t)\ e^{iH_0 t} = \dot{\rho}(t). \tag{4.15}$$

Performing the adiabatic approximation in (4.9) and inserting the resulting expression

$$\begin{aligned} \dot{\rho}^I(t) &= -[V_T^I(t), [A_T, \rho^I(t)]] \\ &= -[e^{iH_0 t}\ V_T(t)\ e^{-iH_0 t}, [A_T, e^{iH_0 t}\ \rho(t)\ e^{-iH_0 t}]] \end{aligned} \tag{4.16}$$

into (4.15), we get the equation of motion for the density operator in the Schrödinger picture

$$\dot{\rho}(t) = -i[H_0, \rho(t)] - [V_T(t), [e^{-iH_0 t}\ A_T\ e^{iH_0 t}, \rho(t)]]. \tag{4.17}$$

For evaluating the term $e^{-iH_0 t}\ A_T\ e^{iH_0 t}$ note that due to the δ-function we can put

$$\begin{aligned} &e^{-iH_0 t}\ \delta(\omega_e + \omega_h + \epsilon_{\bar{\sigma}} - \epsilon_3)\ C_\sigma^\dagger(\omega_e)\ D^\dagger(\omega_h)\ |\bar{\sigma}\rangle\langle 3|\ e^{iH_0 t} \\ &= \delta(\omega_e + \omega_h + \epsilon_{\bar{\sigma}} - \epsilon_3)\ C_\sigma^\dagger(\omega_e)\ D^\dagger(\omega_h)\ |\bar{\sigma}\rangle\langle 3| \end{aligned} \tag{4.18}$$

(i.e., the energies in the oscillatory terms sum up to zero).

In what follows we shall assume a fast relaxation of the hole in the reservoir. This can be achieved if the field-effect structure is designed such that the hole enters with sufficiently high excess energy into the reservoir and immediately suffers an inelastic scattering. In turn, it stays in the reservoir and cannot tunnel back to the dot. Within our approach, this assumption implies that we neglect all terms which would give expressions of the form

$$tr_h\{\rho D^\dagger D\} \simeq 0, \tag{4.19}$$

i.e., the expectation value of the hole number operator $N_h = D^\dagger D$ is taken as zero in all our calculations.

Calculating (4.17) without executing the commutator and neglecting terms of the form CC, $C^\dagger C^\dagger$,DD, $D^\dagger D^\dagger$, we get the equation of motion for the density operator in the Schrödinger picture

$$\dot\rho(t) = -i[H_0,\rho(t)] - \sum_{\sigma\sigma'} \int d\omega_e d\omega_h d\omega'_e d\omega'_h \ \pi \ \delta(\omega_e + \omega_h + \epsilon_{\bar\sigma} - \epsilon_3) \ \times$$

$$\begin{array}{ll} [& \hat t(\omega')C^\dagger_{\sigma'}(\omega'_e) \ D^\dagger(\omega'_h) \ |\bar\sigma'\rangle\langle 3| \ \hat t^*(\omega) \ D(\omega_h) \ C_\sigma(\omega_e) \ |3\rangle\langle\bar\sigma| \ \rho(t) \\ + & \hat t^*(\omega') \ D(\omega'_h) \ C_{\sigma'}(\omega'_e) \ |3\rangle\langle\bar\sigma'|\hat t(\omega) \ C^\dagger_\sigma(\omega_e) \ D^\dagger(\omega_h) \ |\bar\sigma\rangle\langle 3| \ \rho(t) \\ - & \hat t(\omega') \ C^\dagger_{\sigma'}(\omega'_e) \ D^\dagger(\omega'_h) \ |\bar\sigma'\rangle\langle 3| \ \rho(t) \ \hat t^*(\omega) \ D(\omega_h) \ C_\sigma(\omega_e) \ |3\rangle\langle\bar\sigma| \\ - & \hat t^*(\omega') \ D(\omega'_h) \ C_{\sigma'}(\omega'_e) \ |3\rangle\langle\bar\sigma'| \ \rho(t) \ \hat t(\omega) \ C^\dagger_\sigma(\omega_e) \ D^\dagger(\omega_h) \ |\bar\sigma\rangle\langle 3| \\ - & \hat t(\omega) \ C^\dagger_\sigma(\omega_e) \ D^\dagger(\omega_h) \ |\bar\sigma\rangle\langle 3| \ \rho(t) \ \hat t^*(\omega') \ D(\omega'_h) \ C_{\sigma'}(\omega'_e) \ |3\rangle\langle\bar\sigma'| \\ - & \hat t^*(\omega) \ D(\omega_h) \ C_\sigma(\omega_e) \ |3\rangle\langle\bar\sigma| \ \rho(t) \ \hat t(\omega') \ C^\dagger_{\sigma'}(\omega'_e) \ D^\dagger(\omega'_h) \ |\bar\sigma'\rangle\langle 3| \\ + & \rho(t)\hat t(\omega) \ C^\dagger_\sigma(\omega_e) \ D^\dagger(\omega_h) \ |\bar\sigma\rangle\langle 3| \ \hat t^*(\omega') \ D(\omega'_h) \ C_{\sigma'}(\omega'_e) \ |3\rangle\langle\bar\sigma'| \\ + & \rho(t) \ \hat t^*(\omega) \ D(\omega_h) \ C_\sigma(\omega_e) \ |3\rangle\langle\bar\sigma| \ \hat t(\omega') \ C^\dagger_{\sigma'}(\omega'_e) \ D^\dagger(\omega'_h) \ |\bar\sigma'\rangle\langle 3| \ ], \end{array} \tag{4.20}$$

where $\hat t(\omega) = \hat t(\omega_e, \omega_h)$.

Tracing over the hole degrees of freedom in the reservoir and using cyclical permutation of D under the trace we get the master equation for the reduced density operator

$$tr_h \ \dot\rho(t) = -i \ tr_h[H_0,\rho(t)] - \sum_{\sigma\sigma'} \int d\omega_e d\omega'_e d\bar\omega \ \pi \ \delta(\omega_e + \bar\omega + \epsilon_{\bar\sigma} - \epsilon_3) \ \times$$

$$\begin{array}{lll} g^h(\bar\omega) & [& \hat t^*(\omega'_e,\bar\omega) \ C_{\sigma'}(\omega'_e) \ |3\rangle\langle\bar\sigma'|\hat t(\omega_e,\bar\omega) \ C^\dagger_\sigma(\omega_e) \ |\bar\sigma\rangle\langle 3| \ \rho_S(t) \\ & - & \hat t(\omega'_e,\bar\omega) \ C^\dagger_{\sigma'}(\omega'_e) \ |\bar\sigma'\rangle\langle 3| \ \rho_S(t) \ \hat t^*(\omega_e,\bar\omega) \ C_\sigma(\omega_e) \ |3\rangle\langle\bar\sigma| \\ & - & \hat t(\omega_e,\bar\omega) \ C^\dagger_\sigma(\omega_e) \ |\bar\sigma\rangle\langle 3| \ \rho_S(t) \ \hat t^*(\omega'_e,\bar\omega) \ C_{\sigma'}(\omega'_e) \ |3\rangle\langle\bar\sigma'| \\ & + & \rho_S(t) \ \hat t^*(\omega_e,\bar\omega) \ C_\sigma(\omega_e) \ |3\rangle\langle\bar\sigma| \ \hat t(\omega'_e,\bar\omega) \ C^\dagger_{\sigma'}(\omega'_e) \ |\bar\sigma'\rangle\langle 3| \ ]. \end{array} \tag{4.21}$$

Here, we have neglected all terms $tr_h\{\rho D^\dagger D\}$ in accordance to our assumption regarding fast hole relaxation. Note that we put

$$\bar\omega = \omega_h = \omega'_h, \tag{4.22}$$

and use [Eq. (4.3) and (4.6)]

$$tr_h\{DD^\dagger\rho\} = tr_h\{D^\dagger\rho D\} = tr_h\{\rho DD^\dagger\} = g^h. \tag{4.23}$$

In the following we introduce the reduced density operator

$$\rho_S(t) = tr_h\ \rho(t), \tag{4.24}$$

where S solely describes the electrons in the dot and contact.

Let us simplify the master equation (4.21). For this purpose we combine the transition matrix elements $\hat{t}$ and $\hat{t}^*$ with the hole density of states $g^h(\bar{\omega})$ to one expression

$$\hat{T}_{\bar{\sigma}}(\omega_e, \omega_e') = \int d\bar{\omega}\ g^h(\bar{\omega})\ \hat{t}^*(\omega_e, \bar{\omega})\ \hat{t}(\omega_e', \bar{\omega})\ \pi\ \delta(\omega_e + \bar{\omega} + \epsilon_{\bar{\sigma}} - \epsilon_3). \tag{4.25}$$

Thus, the master equation for the reduced density operator $\rho_S(t)$ becomes

$$\begin{aligned}
\dot{\rho}_S(t) = -i\ tr_h[H_0, \rho(t)] - \sum_{\sigma\sigma'} \int d\omega_e d\omega_e'\ \hat{T}_{\bar{\sigma}}(\omega_e, \omega_e')\ \times \\
[\ \ & C_{\sigma'}(\omega_e')\ |3\rangle\langle\bar{\sigma}'|\ C_\sigma^\dagger(\omega_e)\ |\bar{\sigma}\rangle\langle 3|\ \rho_S(t) \\
-\ \ & C_{\sigma'}^\dagger(\omega_e')\ |\bar{\sigma}'\rangle\langle 3|\ \rho_S(t)\ C_\sigma(\omega_e)\ |3\rangle\langle\bar{\sigma}| \\
-\ \ & C_\sigma^\dagger(\omega_e)\ |\bar{\sigma}\rangle\langle 3|\ \rho_S(t)\ C_{\sigma'}(\omega_e')\ |3\rangle\langle\bar{\sigma}'| \\
+\ \ & \rho_S(t)\ C_\sigma(\omega_e)\ |3\rangle\langle\bar{\sigma}|\ C_{\sigma'}^\dagger(\omega_e')\ |\bar{\sigma}'\rangle\langle 3|\ ].
\end{aligned} \tag{4.26}$$

Our next step will be to bring this master equation to the Lindblad form

$$\dot{\rho_S} = -i[H_S, \rho_S] - \frac{1}{2}\sum_i f_i\big(L_i^\dagger L_i \rho_S + \rho_S L_i^\dagger L_i\big) + \sum_i f_i L_i \rho_S L_i^\dagger. \tag{4.27}$$

The sum runs over the various scattering channels i of the system, whose scattering strength is given by f_i.

4.1.2 Determination of the Linblad Operators

We start by considering the scattering channel for tunneling. Comparison of the master equation (4.26) with the general form (4.27) reveals that our Lindblad operators are

$$L = \sum_{\sigma} \int d\omega_e \; C_{\sigma}^{\dagger}(\omega_e) \; |\bar{\sigma}\rangle\langle 3| \tag{4.28}$$

and

$$L^{\dagger} = \sum_{\sigma} \int d\omega_e \; C_{\sigma}(\omega_e) \; |3\rangle\langle\bar{\sigma}|. \tag{4.29}$$

The effective Hamiltonian is described by the system's Hamiltonian H_S plus the non-Hermitian part

$$H_{\text{eff}} = H_S - \frac{i}{2} f \; L^{\dagger} L. \tag{4.30}$$

Apparently, the function f is twice the expression (4.25), i.e.,

$$f = 2 \; \hat{T}_{\bar{\sigma}}(\omega_e, \omega_e'). \tag{4.31}$$

Finally, the system's Hamiltonian H_S is the sum of the QD Hamiltonian, Eq. (3.2), and the part of the reservoir Hamiltonian, Eq. (3.3), for electrons

$$H_S = H_0^D + \sum_{k\sigma} \epsilon_{k\sigma}^{e} c_{k\sigma}^{\dagger} c_{k\sigma}. \tag{4.32}$$

4.1.3 Unraveling of the System's Master Equation

The aim of this section is the solution of the master equation, which we achieve by means of unraveling.

We start with the initial state of a charged exciton in the dot, $|3\rangle$, and no additional electron in the reservoir, $|0\rangle_R$,

$$|\psi_0\rangle = |3\rangle \otimes |0\rangle_R = |3\rangle|0\rangle_R = |0\rangle_R|3\rangle. \tag{4.33}$$

Acting with the Lindblad operator from the left-hand side gives

$$\begin{aligned} L\,|\psi_0\rangle &= \sum_\sigma \int d\omega_e C^\dagger_\sigma(\omega_e)|\bar\sigma\rangle\langle 3|3\rangle|0\rangle_R \\ &= \sum_\sigma \int d\omega_e C^\dagger_\sigma(\omega_e)|\bar\sigma\rangle|0\rangle_R. \end{aligned} \tag{4.34}$$

Further action of $\frac{1}{2}\, f\, L^\dagger$ on the last expression and use of the relation

$$\begin{aligned} C_\sigma(\omega'_e)\; C^\dagger_\sigma(\omega_e)\;|0\rangle_R &= \sum_{k'k} c_{k'\sigma}\delta(\omega'_e - \epsilon^e_{k'\sigma})\; c^\dagger_{k\sigma}\delta(\omega_e - \epsilon^e_{k\sigma})\;|0\rangle_R \\ &= \sum_k \;\delta(\omega'_e - \omega_e)\;\delta(\omega_e - \epsilon^e_{k\sigma})\;|0\rangle_R \\ &= \delta(\omega'_e - \omega_e)\; g^e_\sigma(\omega_e)\;|0\rangle_R, \end{aligned} \tag{4.35}$$

then results in

$$\begin{aligned} \frac{1}{2}\, f\, L^\dagger\, L\,|\psi_0\rangle &= \sum_{\sigma\sigma'} \int d\omega_e\; d\omega'_e\; \hat{T}_{\bar\sigma}(\omega_e,\omega'_e)\; C_{\sigma'}(\omega'_e)|3\rangle\langle\bar\sigma'|\bar\sigma\rangle C^\dagger_\sigma(\omega_e)\;|0\rangle_R \\ &= \sum_\sigma \int d\omega_e\; d\omega'_e\; \hat{T}_{\bar\sigma}(\omega_e,\omega'_e)\;|3\rangle\; C_\sigma(\omega'_e)\; C^\dagger_\sigma(\omega_e)\;|0\rangle_R \\ &= \sum_\sigma \int d\omega_e\; \hat{T}_{\bar\sigma}(\omega_e,\omega_e)\;|3\rangle\; g^e_\sigma(\omega_e)\;|0\rangle_R \\ &= \frac{1}{2}\sum_\sigma \lambda_\sigma\;|3\rangle|0\rangle_R, \end{aligned} \tag{4.36}$$

with the abbreviation λ_σ

$$\lambda_\sigma = 2\int d\omega_e\; g^e_\sigma(\omega_e)\;\hat{T}_{\bar\sigma}(\omega_e,\omega_e). \tag{4.37}$$

The probability for a 'jump' (i.e., tunneling) is the squared norm of the state vector. Therefore the probability for scattering in the time interval Δt is

$$\Delta P = \Delta t\;\langle 3|\;{}_R\langle 0|\; f\; L^\dagger L\;|0\rangle_R|3\rangle = \Delta t \sum_\sigma \lambda_\sigma \equiv \Delta t\;\lambda, \tag{4.38}$$

with $\sum_\sigma \lambda_\sigma = \lambda$ the total scattering rate for tunneling. The probability that no scattering occurs in time interval $[0,t]$ is the complement $1 - \Delta P$. Let us discretize the time interval $[0, t]$ into n steps $\frac{t}{n}$, and let the number of steps approach infinity.

With the help of the relation

$$\lim_{n\to\infty} \big(1+\frac{a}{n}\big)^n = e^a \tag{4.39}$$

the equation for the probability for no scattering becomes

$$\lim_{n\to\infty} \big(1-\frac{t}{n}\sum_\sigma \lambda_\sigma\big)^n = e^{-t\sum_\sigma \lambda_\sigma} = P_0(t). \tag{4.40}$$

Eq. (4.40) gives the probability that up to time t no scattering has occurred. But what is the probability $p(t)$ that a scattering occurs at time t? The answer is provided by

$$p(t) = \frac{d}{dt}(1-P_0(t)) = \sum_\sigma \lambda_\sigma e^{-t\sum_\sigma \lambda_\sigma} = \lambda e^{-t\lambda}. \tag{4.41}$$

This can be proven through

$$\int_0^t dt' p(t') = \int_0^t dt' \lambda e^{-t'\lambda} = -e^{-t'\lambda}\big|_0^t = 1 - P_0(t). \tag{4.42}$$

With these results we are now able to determine the time evolution of the density operator subject to tunneling:

$$\rho_S(t) = P_0(t)\rho_0 + \lambda \int_0^t d\bar{t}\; P_0(\bar{t})\; U(t,\bar{t})\; \rho_1\; U(\bar{t},t). \tag{4.43}$$

The first term describes the conditional density operator for no scattering, with the time-dependent probability $P_0(t) = e^{-t\sum_\sigma \lambda_\sigma} = e^{-t\lambda}$. In other words, this term describes how the probability for the initial state $|\psi_0\rangle\langle\psi_0|$ decays with time.

The second term is the scattering term. This part exhibits the time evolution of the initial density operator under the influence of the tunnel process. $U(t,\bar{t})$ is the time evolution operator for the free propagation of the system due to H_S when no further interactions with the reservoir occur; more realisticly, the system propagates in presence of scatterings in the reservoir, described by $U(t,\bar{t})$. The explicit form of ρ_1, which is the density operator immediately after the tunneling process, is

$$\rho_1 = \frac{f L \rho_0 L^\dagger}{tr\{f L \rho_0 L^\dagger\}}. \tag{4.44}$$

Evaluating the denominator of ρ_1 gives (note that $tr\{L\rho_0 L^\dagger\} = tr\{L^\dagger L \rho_0\} = tr\{\rho_0 L L^\dagger\}$)

$$\begin{aligned} tr\{f L \rho_0 L^\dagger\} &= tr\{2 \sum_{\sigma\sigma'} \int d\omega_e d\omega_e' \; \hat{T}_{\bar{\sigma}}(\omega_e, \omega_e') \times \\ &\quad |\bar{\sigma}'\rangle C^\dagger_{\sigma'}(\omega_e')|0\rangle_R \; {}_R\langle 0| C_\sigma(\omega_e) \langle\bar{\sigma}|\} \\ &= 2 \sum_{\sigma} \int d\omega_e d\omega_e' \; \hat{T}_{\bar{\sigma}}(\omega_e, \omega_e') \; tr\{C^\dagger_\sigma(\omega_e')|0\rangle_R \; {}_R\langle 0|C_\sigma(\omega_e)\} \\ &= \lambda. \end{aligned} \tag{4.45}$$

Finally, the density operator describing our system under the influence of the tunnel process, reads

$$\begin{aligned} \rho_S(t) &= e^{-\lambda t}|3\rangle|0\rangle_R \; {}_R\langle 0|\langle 3| + \lambda \int_0^t d\bar{t} \; e^{-\lambda \bar{t}} \sum_{\sigma\sigma'} \int d\omega_e d\omega_e' \; \frac{\hat{T}_{\bar{\sigma}}(\omega_e, \omega_e')}{\lambda} \times \\ &\quad U(t,\bar{t}) \; |\bar{\sigma}'\rangle \; \otimes C^\dagger_{\sigma'}(\omega_e') \; |0\rangle_R \; {}_R\langle 0| \; C_\sigma(\omega_e) \otimes \langle\bar{\sigma}| \; U(\bar{t},t). \end{aligned} \tag{4.46}$$

Most importantly, the term

$$\sum_{\sigma'} |\bar{\sigma}'\rangle \otimes C^\dagger_{\sigma'}(\omega_e') \; |0\rangle_R = \sum_{\sigma'} |\bar{\sigma}'\rangle \otimes |\sigma'\rangle_R \tag{4.47}$$

reflects the spin entanglement of the electron in the dot-state $|\bar{\sigma}'\rangle$ and the electron in the reservoir-state $C^\dagger_{\sigma'}(\omega_e') \; |0\rangle_R = |\sigma'\rangle_R$ with spin orientation σ', (see discussion in Appendix C).

We see that the tunnel process leads to entanglement of the electrons in the dot and reservoir, respectively. Thus our first point in Chapter 3 is answered.

4.2 First Scattering in the Reservoir

The considerations in the previous sections will help us to answer our second question regarding the influence of spin-unselectiv scattering in the reservoir on the spin entanglement.

First we choose our Lindblad operators in a convenient way

$$L(\omega_e) = \sqrt{\Gamma}\,\sum_\sigma \frac{C^\dagger_\sigma(\omega_e)C_\sigma(\omega_e)}{g^e_\sigma(\omega_e)} = L^\dagger(\omega_e), \tag{4.48}$$

with Γ the scattering rate, and $C^\dagger_\sigma(\omega_e)C_\sigma(\omega_e)$ the number operator for electrons with spin σ and energy ω_e. Eq. (4.48) is a simplified description model that accounts for elastic scattering processes (e.g., due to phonons). Despite its simplicity we believe that it is general enough to answer our question regarding decoherence, and that more sophisticated interaction models would not drastically alter the conclusions of this section.

The initial state is now the entangled state appearing in (4.46)

$$|\phi\rangle = \sum_\sigma \int d\omega_e C^\dagger_\sigma(\omega_e)\; |\bar{\sigma}\rangle \otimes |0\rangle_R. \tag{4.49}$$

The vector is not normalized, and the scalar product

$$\begin{aligned}
\langle\phi|\phi\rangle &= \sum_{\sigma\sigma'} \int d\omega_e d\omega'_e\; \langle\bar{\sigma}|\bar{\sigma}'\rangle\; {}_R\langle 0|C_{\sigma'}(\omega'_e)\; C^\dagger_\sigma(\omega_e)|0\rangle_R \\
&= \sum_\sigma \int d\omega_e d\omega'_e\; {}_R\langle 0|C_\sigma(\omega'_e)\; C^\dagger_\sigma(\omega_e)|0\rangle_R \\
&= \sum_\sigma \int d\omega_e\; g^e_\sigma(\omega_e),
\end{aligned}\tag{4.50}$$

with

$${}_R\langle 0|C_\sigma(\omega'_e)\; C^\dagger_\sigma(\omega_e)|0\rangle_R = \delta(\omega'_e - \omega_e) g^e_\sigma(\omega_e), \tag{4.51}$$

provides the appropriate norm factor. The normalized initial state vector then reads

$$|\phi_0\rangle = \sum_\sigma \int d\omega_e\; (2\; g^e_\sigma(\omega_e))^{-\frac{1}{2}}\; C^\dagger_\sigma(\omega_e)\; |\bar{\sigma}\rangle \otimes |0\rangle_R. \tag{4.52}$$

Remember that the initial state vector for the conditional evolution of 'no jumps' evolves in the unraveling-method of the master equation with the effective Hamiltonian

$$H_{\rm eff} = H_S - \frac{i}{2}\int d\omega L^{\dagger}(\omega)\; L(\omega), \tag{4.53}$$

where we have integrated over all possible scattering channels with energy ω.

The product of the two Lindblad operators for the first scattering in the reservoir is

$$L^{\dagger}(\omega_e)\; L(\omega_e) = \Gamma \sum_{\sigma\sigma'} \frac{1}{g^e_{\sigma}(\omega_e) g^e_{\sigma'}(\omega'_e)} C^{\dagger}_{\sigma}(\omega_e) C_{\sigma}(\omega_e) C^{\dagger}_{\sigma'}(\omega'_e) C_{\sigma'}(\omega'_e). \tag{4.54}$$

Acting with the effective Hamiltonian $H_{\rm eff}$ on the entangled state $|\phi_0\rangle$

$$\begin{aligned} H_{\rm eff}|\phi_0\rangle &= \sum_{\sigma}\int d\omega_e (2\, g^e_{\sigma}(\omega_e))^{-\frac{1}{2}}\; \{(\epsilon_{\bar\sigma} + \omega_e) C^{\dagger}_{\sigma}(\omega_e)\; |\bar\sigma\rangle \otimes |0\rangle_R \\ &- \frac{i\Gamma}{2} \sum_{ss'} \int d\bar\omega_e d\bar\omega'_e\; \frac{1}{g^e_s(\bar\omega_e) g^e_{s'}(\bar\omega'_e)} \times \\ &\quad C^{\dagger}_s(\bar\omega_e) C_s(\bar\omega_e) C^{\dagger}_{s'}(\bar\omega'_e) C_{s'}(\bar\omega'_e)\; C^{\dagger}_{\sigma}(\omega_e)\; |\bar\sigma\rangle \otimes |0\rangle_R\}, \end{aligned} \tag{4.55}$$

terms arise which can be simplified by using

$$C_{s'}(\bar\omega'_e)\; C^{\dagger}_{\sigma}(\omega_e)\; |0\rangle_R = \delta_{s'\sigma}\delta(\omega_e - \bar\omega'_e)\; g^e_{\sigma}(\omega_e)\; |0\rangle_R$$

and

$$C_s(\bar\omega_e) C^{\dagger}_{s'}(\bar\omega'_e)\; |0\rangle_R = \delta_{ss'}\delta(\bar\omega_e - \bar\omega'_e)\; g^e_s(\bar\omega_e)\; |0\rangle_R. \tag{4.56}$$

This yields to

$$H_{\rm eff}|\phi_0\rangle = \sum_{\sigma}\int d\omega_e\; (2\, g^e_{\sigma}(\omega_e))^{-\frac{1}{2}} \left(\epsilon_{\bar\sigma} + \omega_e - \frac{i\Gamma}{2}\right) C^{\dagger}_{\sigma}(\omega_e)\; |\bar\sigma\rangle \otimes |0\rangle_R. \tag{4.57}$$

We next assume that the electron energy is independent of the spin orientation $\epsilon_{\sigma} = \epsilon$, i.e, the energy for the electron with spin-up is the same as for spin-down. Thus, Eq. (4.57) gets

$$H_{\rm eff}|\phi_0\rangle \simeq \sum_{\sigma}\int d\omega_e\; (2\, g^e_{\sigma}(\omega_e))^{-\frac{1}{2}} \left(\epsilon + \omega_e - \frac{i\Gamma}{2}\right) C^{\dagger}_{\sigma}(\omega_e)\; |\bar\sigma\rangle \otimes |0\rangle_R. \tag{4.58}$$

The next task is to evaluate the probability for a jump, i.e., the elastic scattering event in the reservoir (again, $(1 - i\ H_{\text{eff}} \Delta t)|\psi\rangle$ accounts for the evolution of the state $|\psi\rangle$ if no jump occurs). For short time intervals

$$1 - i\ H_{\text{eff}} \Delta t \simeq e^{-i\ H_{\text{eff}} \Delta t}, \tag{4.59}$$

and the conditional density operator evolves as

$$\begin{aligned} |\tilde{\phi}_0\rangle\langle\tilde{\phi}_0| &= e^{-i\ H_{\text{eff}} t}\ |\phi_0\rangle\langle\phi_0|\ e^{i\ H_{\text{eff}}^\dagger t} \\ &\simeq \sum_{\sigma\sigma'} \int d\omega_e d\omega_e'\ \left(4\ g_\sigma^e(\omega_e) g_{\sigma'}^e(\omega_e')\right)^{-\frac{1}{2}}\ e^{-i(\omega_e - \omega_e' - i\Gamma)t} \times \\ &\quad C_\sigma^\dagger(\omega_e)\ |\bar{\sigma}\rangle \otimes |0\rangle_R\ {}_R\langle 0| \otimes \langle\bar{\sigma}'|\ C_{\sigma'}(\omega_e'). \end{aligned} \tag{4.60}$$

The probability of this configuration decays as

$$\begin{aligned} tr\{|\tilde{\phi}_0\rangle\langle\tilde{\phi}_0|\} &\simeq \sum_{\sigma} \int d\omega_e d\omega_e' \left(4\ g_\sigma^e(\omega_e) g_\sigma^e(\omega_e')\right)^{-\frac{1}{2}}\ e^{-i(\omega_e - \omega_e' - i\Gamma)t} \times \\ &\quad tr\{C_\sigma^\dagger(\omega_e)\ |0\rangle_R\ {}_R\langle 0|\ C_\sigma(\omega_e')\}. \end{aligned} \tag{4.61}$$

Using

$$tr\{C_\sigma^\dagger(\omega_e)\ |0\rangle_R\ {}_R\langle 0|\ C_\sigma(\omega_e')\} = g_\sigma^e(\omega_e)\delta(\omega_e - \omega_e') \tag{4.62}$$

we obtain

$$tr\{|\tilde{\phi}_0\rangle\langle\tilde{\phi}_0|\} \simeq e^{-\Gamma t}. \tag{4.63}$$

This is the probability for the conditional evolution of no jumps. As expected, it leads to a decay of the initial density operator in time, and with the scattering rate Γ.

For one scattering channel (energy ω_e) the action of the Lindblad operator on the entangled state reads

$$\begin{aligned} L(\omega_e)|\phi_0\rangle &= \sqrt{\Gamma} \sum_{\sigma\sigma'} \frac{C_\sigma^\dagger(\omega_e) C_\sigma(\omega_e)}{g_\sigma^e(\omega_e)} \int d\omega_e' \left(2\ g_{\sigma'}^e(\omega_e')\right)^{-\frac{1}{2}} \times \\ &\quad C_{\sigma'}^\dagger(\omega_e')\ |\bar{\sigma}'\rangle \otimes |0\rangle_R \\ &= \sum_\sigma \left(\frac{2\ g_\sigma^e(\omega_e)}{\Gamma}\right)^{-\frac{1}{2}}\ C_\sigma^\dagger(\omega_e)\ |\bar{\sigma}\rangle \otimes |0\rangle_R. \end{aligned} \tag{4.64}$$

The corresponding density operator is

$$\begin{aligned}|\phi_1\rangle\langle\phi_1| &= \int d\omega_e\ L(\omega_e)|\phi_0\rangle\langle\phi_0|L(\omega_e)^\dagger \\ &= \sum_{\sigma\sigma'}\int d\omega_e\ \Gamma\ \left(4\ g^e_\sigma(\omega_e)g^e_{\sigma'}(\omega_e)\right)^{-\frac{1}{2}} \times \\ &\quad C^\dagger_\sigma(\omega_e)\ |\bar{\sigma}\rangle\otimes|0\rangle_R\ {}_R\langle 0|\otimes\langle\bar{\sigma}'|\ C_{\sigma'}(\omega_e). \qquad (4.65)\end{aligned}$$

As a result of the first scattering in the reservoir, we observe that the term $C^\dagger_\sigma(\omega_e)\ |\bar{\sigma}\rangle\langle\bar{\sigma}'|\ C_{\sigma'}(\omega_e)$ becomes diagonal in energy (i.e., dephasing), whereas the spin components are not affected. With this our second question in Chapter 3 is answered.

With the help of

$$\begin{aligned}&tr\ \ \{C^\dagger_\sigma(\omega_e)\ |\bar{\sigma}\rangle|0\rangle_R\ {}_R\langle 0|\langle\bar{\sigma}'|\ C_{\sigma'}(\omega_e)\} \\ =&\ \sum_s\langle s|\bar{\sigma}\rangle\langle\bar{\sigma}'|s\rangle\ tr\{C^\dagger_\sigma(\omega_e)\ |0\rangle_R\ {}_R\langle 0|\ C_{\sigma'}(\omega_e)\} \\ =&\ \sum_s\delta_{s\bar{\sigma}}\ \delta_{s\bar{\sigma}'}\ g^e_\sigma(\omega_e)\ \delta_{\sigma\sigma'} \qquad (4.66)\end{aligned}$$

we get the norm of the operator $|\phi_1\rangle\langle\phi_1|$

$$tr\{|\phi_1\rangle\langle\phi_1|\} = \Gamma. \qquad (4.67)$$

The probability for scattering in the time interval Δt is

$$\Delta P = \Delta t\ \langle\phi_0|L(\omega_e)^\dagger L(\omega_e)|\phi_0\rangle \qquad (4.68)$$

Using (4.56) and the relation

$$L(\omega_e)|\phi_0\rangle = \sum_\sigma\left(\frac{2\ g^e_\sigma(\omega_e)}{\Gamma}\right)^{-\frac{1}{2}}\ C^\dagger_\sigma(\omega_e)\ |\bar{\sigma}\rangle\otimes|0\rangle_R \qquad (4.69)$$

we get

$$\Delta P = \Delta t\ \Gamma. \qquad (4.70)$$

We see that each scattering channel has the same probability, which is due to our simplified description of scatterings in the environment. The probability for no scattering in the time interval $[0,t]$ is

$$\lim_{n\to\infty}(1-\frac{t}{n}\Gamma)^n = e^{-\Gamma t} = P_{0,\Gamma}(t). \qquad (4.71)$$

Finally, with the help of

$$(1 - P_{0,\Gamma}(t)) = \int_0^t d\bar{t}\ \Gamma\ e^{-\Gamma \bar{t}}, \tag{4.72}$$

we express the conditional density operator as

$$\tilde{\rho}(t) = e^{-\Gamma t} \frac{|\tilde{\phi}_0\rangle\langle\tilde{\phi}_0|}{tr\{|\tilde{\phi}_0\rangle\langle\tilde{\phi}_0|\}} + \Gamma \int_0^t d\bar{t}\ e^{-\Gamma \bar{t}}\ \tilde{U}(t,\bar{t}) \frac{|\phi_1\rangle\langle\phi_1|}{tr\{|\phi_1\rangle\langle\phi_1|\}} \tilde{U}(\bar{t},t). \tag{4.73}$$

We can identify the first term as a part for no scattering with probability $e^{-\Gamma t}$, and the second term as a part for scattering with probability $\int_0^t d\bar{t}\ \Gamma\ e^{-\Gamma \bar{t}} = 1 - e^{-\Gamma t}$.

Putting together all our results we obtain

$$\begin{aligned} \tilde{\rho}(t) \quad = \quad & e^{-\Gamma t} \sum_{\sigma\sigma'} \int d\omega_e d\omega'_e\ \left(4\ g^e_\sigma(\omega_e) g^e_{\sigma'}(\omega'_e)\right)^{-\frac{1}{2}}\ e^{-i(\omega_e - \omega'_e)t} \times \\ & C^\dagger_\sigma(\omega_e)\ |\bar{\sigma}\rangle \otimes |0\rangle_R\ {}_R\langle 0| \otimes \langle\bar{\sigma}'|\ C_{\sigma'}(\omega'_e) \\ + \quad & \Gamma \int_0^t d\bar{t}\ e^{-\Gamma \bar{t}}\ \tilde{U}(t,\bar{t}) \sum_{\sigma\sigma'} \int d\omega_e\ \left(4\ g^e_\sigma(\omega_e) g^e_{\sigma'}(\omega_e)\right)^{-\frac{1}{2}} \times \\ & C^\dagger_\sigma(\omega_e)\ |\bar{\sigma}\rangle \otimes |0\rangle_R\ {}_R\langle 0| \otimes \langle\bar{\sigma}'|\ C_{\sigma'}(\omega_e)\ \tilde{U}(\bar{t},t). \end{aligned} \tag{4.74}$$

This is the expression for the density operator due to the elastic spin-unselective scattering event in the reservoir. Next, we replace

$$\sum_{\sigma\sigma'} \int d\omega_e d\omega'_e\ \frac{\hat{T}_{\bar{\sigma}}(\omega_e, \omega'_e)}{\lambda}\ U(t,\bar{t})|\bar{\sigma}'\rangle\ C^\dagger_{\sigma'}(\omega'_e)\ |0\rangle_R\ {}_R\langle 0|\ C_\sigma(\omega_e)\langle\bar{\sigma}|\ U(\bar{t},t)$$

from equation (4.46) by the density operator (4.74). Performing this replacement we get the somewhat lengthy expression

$$\begin{aligned}
\rho_S(t) &= e^{-\lambda t}|3\rangle|0\rangle_R\ {}_R\langle 0|\langle 3| + \lambda \int_0^t dt'\ e^{-\lambda t'}\ \tilde{\rho}(t-t') \\
&= e^{-\lambda t}|3\rangle|0\rangle_R\ {}_R\langle 0|\langle 3| + \lambda \int_0^t dt'\ e^{-\lambda t'} \times \\
& \quad \Big\{ e^{-\Gamma(t-t')} \sum_{\sigma\sigma'} \int d\omega_e d\omega'_e\ \left(4\ g^e_\sigma(\omega_e) g^e_{\sigma'}(\omega'_e)\right)^{-\frac{1}{2}}\ e^{-i(\omega_e-\omega'_e)(t-t')} \times \\
& \quad C^\dagger_\sigma(\omega_e)\ |\bar{\sigma}\rangle \otimes |0\rangle_R\ {}_R\langle 0| \otimes \langle \bar{\sigma}'|\ C_{\sigma'}(\omega'_e) \\
&+ \Gamma \int_{t'}^t d\bar{t}\ e^{-\Gamma\bar{t}}\ \tilde{U}(t,\bar{t}) \sum_{\sigma\sigma'} \int d\omega_e\ \left(4\ g^e_\sigma(\omega_e) g^e_{\sigma'}(\omega_e)\right)^{-\frac{1}{2}} \times \\
& \quad C^\dagger_\sigma(\omega_e)\ |\bar{\sigma}\rangle \otimes |0\rangle_R\ {}_R\langle 0| \otimes \langle \bar{\sigma}'|\ C_{\sigma'}(\omega_e)\ \tilde{U}(\bar{t},t) \Big\}. \qquad (4.75)
\end{aligned}$$

The first term accounts for the damping of our initial density operator (describing the charged exciton and no additional electron in the reservoir) due to tunneling. The second term is the conditional density operator for the system after tunneling but before the first scattering in the reservoir. The third term is the conditional density operator after scattering, and responsible for the loss of coherence[3] in our system. Note that the second and third term describe both spin entanglement.

4.3 Spin Relaxation in the Reservoir

To investigate the disentanglement due to spin scatterings in the reservoir we will introduce the Pauli spin matrices. Because we have assumed equal scattering strengths for all interaction channels, in the following we are allowed to integrate over all electron energies viz

$$\int d\omega_e (g^e_\sigma(\omega_e) g^e_{\sigma'}(\omega_e))^{-\frac{1}{2}} C^\dagger_\sigma(\omega_e)\ |0\rangle_R\ {}_R\langle 0|\ C_{\sigma'}(\omega_e) \equiv \mathcal{P}_{\sigma\sigma'}. \qquad (4.76)$$

The initial density operator for our system S after decoherence thus has the form

$$|\xi_0\rangle\langle\xi_0| = \frac{\sum_{\sigma\sigma'} \mathcal{P}_{\sigma\sigma'} |\bar{\sigma}\rangle\langle\bar{\sigma}'|}{tr\{\sum_{\sigma\sigma'} \mathcal{P}_{\sigma\sigma'} |\bar{\sigma}\rangle\langle\bar{\sigma}'|\}}. \qquad (4.77)$$

---

[3] As mentioned before, Eq. (4.2), only dephasing of the orbital degrees of freedom occurs (and not of the spin ones).

Remind that $\mathcal{P}_{\sigma\sigma'}$ acts on the reservoir only, whereas $|\bar{\sigma}\rangle\langle\bar{\sigma}'|$ acts only on the dot system.

With the help of the Pauli spin matrices (for the detailed calculation see Appendix D) we get

$$\begin{aligned}\dot{\rho} &= 2\big\{2u_0(\vec{a}' \times \vec{a}'') - (a_0' \, \vec{a}'' - a_0'' \, \vec{a}') \times \vec{u} \\ &\quad -[|\vec{a}|^2 \, \vec{u} - (\vec{a}' \, \vec{u}) \, \vec{a}' - (\vec{a}'' \, \vec{u}) \, \vec{a}'']\big\} \, \vec{\sigma}, \end{aligned} \tag{4.78}$$

where $\rho = u_0 \mathbf{1} + \vec{u} \, \vec{\sigma}$. It follows that[4]

$$\dot{u}_0 = \frac{1}{2} tr(.) = 0, \tag{4.79}$$

where $tr(.)$ means the trace of the right-hand side of Eq. (4.78) .

This is a crucial result which shows that spin scatterings in the reservoir cannot affect the population in the dot.

Consider now the Lindblad operator

$$L = \sqrt{\Gamma}(\mathbf{1} + \kappa\sigma_3), \tag{4.80}$$

where Γ denotes a phenomenological damping constant, κ is a scalar factor, and σ_3 denotes the Pauli matrix. According to Eq. (D.13) we identify that $a_0 = \sqrt{\Gamma}$, and $\vec{a} = \sqrt{\Gamma} \; \kappa \; \hat{e}_3$, where $\hat{e}_3$ is the unit vector in z-direction. With this choice the vector $\vec{a}$ and the scalar factor a_0 only have real parts

$$\begin{aligned} &\vec{a} = \vec{a}' \\ &\vec{a}'' = 0 \\ &a_0'' = 0. \end{aligned} \tag{4.81}$$

To calculate the density operator we have to evaluate

$$\frac{d}{dt}(\vec{u} \, \vec{\sigma}) \;=\; -2 \, [|\vec{a}|^2 \, \vec{u} - (\vec{a}' \, \vec{u}) \, \vec{a}'] \, \vec{\sigma}, \tag{4.82}$$

---

[4] Note that there is no $\mathbf{1}$ on the right-hand side of equation (4.78), and the trace of the unit operator is 2.

where $\vec{u} = u_1\hat{e}_1 + u_2\hat{e}_2 + u_3\hat{e}_3$. It follows that

$$\dot{u}_{1,2} = -2\Gamma\ \kappa^2\ u_{1,2} \tag{4.83}$$

and

$$\dot{u}_3 = 0. \tag{4.84}$$

Integrating (4.83) with respect to time gives

$$u_{1,2}(t) = u_{1,2}(0)\ e^{-2\kappa^2\ \Gamma t}. \tag{4.85}$$

The density operator (4.77) after spin-selective scattering in reservoir has thus the final form

$$\rho(t) = \frac{1}{2}\{|+\rangle\langle+|\mathcal{P}_{++} + |-\rangle\langle-|\mathcal{P}_{--} + e^{-2\kappa^2\ \Gamma t}[|+\rangle\langle-|\mathcal{P}_{+-} + |-\rangle\langle+|\mathcal{P}_{-+}]\}. \tag{4.86}$$

The first two terms show that the diagonal terms, i.e., the populations of the density operator, do not change, whereas an exponential decay of the coherence terms leads to disentanglement. Thus, in our calculations any spin-selective scattering forces the spins with equal probability to one of the two orientations. As a consequence, to experimentally detect spin entanglement in the proposed scheme both electrons have to be monitored.

Chapter 5

Summary

In conclusion, we have proposed a scheme for an optically triggered spin entanglement of electrons in semiconductors. It consists of a single-electron doped quantum dot embedded in a field-effect structure. Optical excitation of an additional electron-hole pair (charged exciton) is transferred through tunneling to a photocurrent, where the spins of the electrons in the dot and reservoir are maximally entangled.

Using a density-operator approach in Lindblad form we have derived analytic expressions for the whole cascade process of: the tunneling decay of the charged exciton to the freely propagating electron and hole in the reservoirs, respectively, and the electron in the dot; dephasing and relaxation of the carriers in the contacts; and finally disentanglement through spin-selective scatterings. This analysis has revealed that the quantum properties of the charged exciton state are transferred through tunneling to the spatially separated carriers in the dot and reservoir, where the spin part of the electronic wavefunction is a maximally entangled state. This entanglement is not destroyed by spin-unselective dephasing and relaxation mechanisms, and is therefore of robust nature.

The proposed device might be useful in future quantum information applications to establish entanglement between spatially separated sites, where one could benefit from the extremely long spin lifetimes of electrons in semiconductors.

Appendix A

Schrödinger Picture vs. Interaction Picture

In this appendix I give a short overview about the relation between the Schrödinger picture and the interaction one.

A.1 Schrödinger Picture

In the Schrödinger picture any state $|\psi(t)\rangle$ evolves in time according to the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle \tag{A.1}$$

which can be formally solved to

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle. \tag{A.2}$$

Here $U(t,t_0)$ is the time evolution operator. If the Hamiltonian H does not depend on time Eq. (A.2) becomes

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}H(t-t_0)}|\psi(t_0)\rangle, \tag{A.3}$$

where

$$U(t,t_0) = e^{-\frac{i}{\hbar}H(t-t_0)}. \tag{A.4}$$

In the Schrödinger picture the states are time-dependent whereas the operators of the physical observables are time-independent (except for a possible explicit time-dependency).

A.2 Interaction Picture

When the Hamiltonian H of the system can be separated into a time-independent part H_0 and a possibly time-dependent perturbation $V(t)$

$$H = H_0 + V(t) \tag{A.5}$$

the use of the interaction picture is convenient. We label the interaction picture with the superscript I, and do not explicitly indicate states and operators in the Schrödinger picture. Thus, we express states and operators through,

$$|\psi(t)\rangle^I = e^{\frac{i}{\hbar}H_0(t-t_0)}|\psi(t)\rangle, \tag{A.6}$$

and[1]

$$\mathcal{A}^I(t) = e^{\frac{i}{\hbar}H_0(t-t_0)}\ \mathcal{A}(t)\ e^{-\frac{i}{\hbar}H_0(t-t_0)}. \tag{A.7}$$

At time t_0 Schrödinger-states coincidence with the interaction ones, $|\psi(t_0)\rangle = |\psi(t_0)\rangle^I$. Without loss of generality we set $t_0 = 0$.

To get the appropriate equation of motion in the interaction picture we have to differentiate (A.6). This gives

$$\frac{\partial}{\partial t}|\psi(t)\rangle^I = e^{\frac{i}{\hbar}H_0 t}(\frac{i}{\hbar}H_0|\psi(t)\rangle + |\dot{\psi}(t)\rangle) \tag{A.8}$$

which can be cast with the Schrödinger equation to

$$\frac{\partial}{\partial t}|\psi(t)\rangle^I = e^{\frac{i}{\hbar}H_0 t}(\frac{i}{\hbar}H_0|\psi(t)\rangle - \frac{i}{\hbar}[H_0 + V(t)]|\psi(t)\rangle). \tag{A.9}$$

Using (A.6) we finally obtain

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle^I = V^I(t)|\psi(t)\rangle^I. \tag{A.10}$$

Note that

$$V^I(t) = e^{\frac{i}{\hbar}H_0 t}\ V(t)\ e^{-\frac{i}{\hbar}H_0 t}. \tag{A.11}$$

---

[1] Note that $e^{\alpha B}\ A\ e^{-\alpha B} = A + \alpha\,[B,A] + \frac{\alpha^2}{2!}\,[B,[B,A]] + \ldots$

The equation of motion for operators in the interaction picture is

$$\frac{d}{dt}\mathcal{A}^I(t) = \frac{i}{\hbar}[H_0, \mathcal{A}^I(t)] + \frac{\partial}{\partial t}\mathcal{A}^I(t). \tag{A.12}$$

As one can see from Eqs. (A.10) and (A.12), the time evolution of the states is governed by $V^I(t)$, and that of the operators by H_0. The last term in (A.12) appears only if the operator $\mathcal{A}$ is explicitly time-dependent.

Appendix B

Direct Product and N-Particle-States

Let v_i^S, $i = 1, ..., N_S$, be elements of a N_S-dimensional vector space (Hilbert space) S and v_j^R, $j = 1, ..., N_R$, elements of a N_R-dimensional vector space R. Then the *direct product* $v_i^S v_j^R$ is neither an element of S nor of R, but spans a $N_S \times N_R$-dimensional space, the product space $G = S \otimes R$.

Consider operators A_S acting in space S (with states $|s\rangle$), and correspondingly operators A_R acting in space R (with states $|r\rangle$). Then the operators $A_S \otimes A_R$ act in the direct product space of the states

$$|s\rangle \otimes |r\rangle = |s, r\rangle = |s\rangle|r\rangle = |r\rangle|s\rangle, \text{with the prescription} \tag{B.1}$$

$$\langle s, r|A_S \otimes A_R|s', r'\rangle = \langle s|A_S|s'\rangle\langle r|A_R|r'\rangle. \tag{B.2}$$

An operator A_S corresponds in this product space to $A_S \otimes I_R$, and A_R corresponds to $A_R \otimes I_S$; here I_i is the unity operator in the space $i = S, R$. Note that

$$A_S B_S \otimes C_R D_R = \big(A_S \otimes C_R\big)\big(B_S \otimes D_R\big). \tag{B.3}$$

Let $|\alpha_i\rangle$, $i = 1, 2, ..., N$, be N orthonormal one-particle states, then a N-particle state may be written as the direct product of these states

$$|\alpha_1, \alpha_2, ..., \alpha_N\rangle = |\alpha_1\rangle|\alpha_2\rangle...|\alpha_N\rangle. \tag{B.4}$$

The ortho-normality relations read

$$\langle \alpha_1, \alpha_2, ...\alpha_N|\alpha_1', \alpha_2'...\alpha_N'\rangle = \delta(\alpha_1, \alpha_1')...\delta(\alpha_N, \alpha_N'). \tag{B.5}$$

Where,

$$\begin{array}{rcl} \delta(\alpha_1, \alpha_1') & \rightarrow & \delta(\alpha_1 - \alpha_1') \\ & \text{i.e.,} & \delta\text{- function, in the case of unproper states, and} \\ \delta(\alpha_1, \alpha_1') & \rightarrow & \delta_{\alpha_1, \alpha_1'} \\ & \text{i.e.,} & \text{Kronecker-Delta, for discrete states.} \end{array} \tag{B.6}$$

The last equation predicates that a set of states $\{|s\rangle\}$ and $\{|r\rangle\}$, which build an orthonormal basis in S and R, respectively, constitutes also an orthonormal basis in the product space $\{|sr\rangle = |s\rangle|r\rangle\}$,

$$\langle sr|s'r'\rangle = \langle s|s'\rangle\langle r|r'\rangle = \delta(s, s')\delta(r, r'). \tag{B.7}$$

Appendix C

Entangled States

An entangled state has no classical analog. The important criterion for an entangled state is the fact that such a state cannot be factorized. For a better understanding I give an example.

Consider a system consisting of two subsystems, e.g., two particles (photons, electrons,...), with state vectors

$$|\psi\rangle_1 = \alpha_1|a\rangle + \beta_1|b\rangle \text{ for particle 1 , and}$$

$$|\psi\rangle_2 = \alpha_2|a\rangle + \beta_2|b\rangle \text{ for particle 2.} \tag{C.1}$$

$\{|a\rangle, |b\rangle\}$ build an orthonormal basis, and α and β correspond to the probability amplitudes to be in state $|a\rangle$ or $|b\rangle$, respectively. If the two particles are uncorrelated then the total state vector is

$$|\chi\rangle = |\psi\rangle_1|\psi\rangle_2 = \alpha_1\alpha_2|a\rangle|a\rangle + \beta_1\beta_2|b\rangle|b\rangle + \alpha_1\beta_2|a\rangle|b\rangle + \beta_1\alpha_2|b\rangle|a\rangle. \tag{C.2}$$

For a correlated state instead we have an entangled state

$$|\chi\rangle = \alpha_1\alpha_2|a\rangle|a\rangle + \beta_1\beta_2|b\rangle|b\rangle \neq |\psi_1\rangle|\psi_2\rangle, \tag{C.3}$$

which cannot be factorized. For a maximum entangled state $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ holds.

The nature of entanglement is such that the properties (such as polarization or momentum) of each individual particle are indeterminate and undefined (until the particle is measured). Measuring one entangled particle, however, determines not only its properties but at the same instant those of its partner (even if they are very far apart). Due to the fact that the two particles are entangled, measurement of the one causes an instantaneous effect on the other. This means that entangled particles cannot be measured independently.

To make a link to our calculations we introduce for the dot and reservoir states of our basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ the relation $\{|0\rangle, |1\rangle\}$

$$|\uparrow\rangle \equiv |0\rangle_1$$

$$|\downarrow\rangle_R \equiv |0\rangle_2$$
$$|\downarrow\rangle \equiv |1\rangle_1$$
$$|\uparrow\rangle_R \equiv |1\rangle_2.$$

According to Eq. (C.3) the normalized maximum entangled spin state of the electron in the dot and reservoir is

$$\begin{aligned} |\chi\rangle &= \frac{1}{\sqrt{2}}\{|0\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2\} \\ &= \frac{1}{\sqrt{2}}\{|00\rangle + |11\rangle\}. \end{aligned} \tag{C.4}$$

Now, going back to the representation $\{|\uparrow\rangle, |\downarrow\rangle\}$ to obtain

$$|\chi\rangle = \frac{1}{\sqrt{2}}\{|\uparrow\ \ \downarrow\rangle + |\downarrow\ \ \uparrow\rangle\} \tag{C.5}$$

for the entangled state in our spin convention.

Appendix D

Pauli Spin Matrices

To derive Eq. (4.78) consider that the spin-up $|+\rangle$ and the spin-down $|-\rangle$ state of our basis can be represented in a matrix notation by the spinor

$$\Sigma_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{D.1}$$

and

$$\Sigma_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{D.2}$$

respectively. The outer product of the two spinors in this representation gives, e.g $\mathcal{P}_{++}$

$$\mathcal{P}_{++} \propto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{D.3}$$

or $\mathcal{P}_{+-}$

$$\mathcal{P}_{+-} \propto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{D.4}$$

The next task is to represent the Pauli spin matrices in terms of $\mathcal{P}$. Because of the restriction

$$\begin{aligned} tr\{\mathcal{P}_{\sigma\sigma'}\} &= tr\{\int d\omega_e (g^e_\sigma(\omega_e) g^e_{\sigma'}(\omega_e))^{-\frac{1}{2}} C^\dagger_\sigma(\omega_e)\ |0\rangle_R\ {}_R\langle 0|\ C_{\sigma'}(\omega_e)\} \\ &= 1 \end{aligned} \tag{D.5}$$

we can put

$$\begin{aligned}\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathcal{P}_{+-} + \mathcal{P}_{-+} \\ \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i(\mathcal{P}_{+-} - \mathcal{P}_{-+}) \\ \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \mathcal{P}_{++} - \mathcal{P}_{--} \\ \mathbf{1} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{P}_{++} + \mathcal{P}_{--}. \end{aligned} \tag{D.6}$$

Note that for Pauli matrices

$$tr\sigma_i = 0 \tag{D.7}$$

holds, and

$$\sigma_i\sigma_j = \delta_{ij}\mathbf{1} + i\varepsilon_{ijk}\sigma_k, \tag{D.8}$$

with ε the fully antisymmetric tensor of third rank. We can write the initial density operator (4.77) thus in the form

$$\begin{aligned}|\xi_0\rangle\langle\xi_0| &= \frac{\sum_{\sigma\sigma'} \mathcal{P}_{\sigma\sigma'}|\sigma\rangle\langle\sigma'|}{tr\{\sum_{\sigma\sigma'} \mathcal{P}_{\sigma\sigma'}|\sigma\rangle\langle\sigma'|\}} \\ &= \frac{1}{4}\left\{|+\rangle\langle+|\,(\mathbf{1}+\sigma_3)\ +\ |-\rangle\langle-|\,(\mathbf{1}-\sigma_3)\right. \\ &+\ \left.|+\rangle\langle-|\,(\sigma_1+i\sigma_2)\ +\ |-\rangle\langle+|\,(\sigma_1-i\sigma_2)\right\}. \end{aligned} \tag{D.9}$$

The most general form for a density operator in terms of Pauli spin matrices is

$$\rho = u_0\mathbf{1} + \vec{u}\ \vec{\sigma}, \tag{D.10}$$

where u_0 is a scalar, $\vec{u}$ describes a vector and $\vec{\sigma}$ is the vector of Pauli matrices. The part of the master equation which is responsible for scatterings is

$$\dot{\rho} = \sum_i \left(L_i\rho L_i^\dagger - \frac{1}{2}[L_i^\dagger L_i, \rho]_+\right), \tag{D.11}$$

where L_i are the generic Lindblad operators acting in the reservoir. Rewriting this equation in terms of commutators

$$[L_i^\dagger L_i, \rho]_+ = L_i^\dagger L_i\rho + \rho L_i^\dagger L_i$$

$$[L_i\rho, L_i^\dagger] + [L_i, \rho L_i^\dagger] = L_i\rho L_i^\dagger - L_i^\dagger L_i\rho + L_i\rho L_i^\dagger - \rho L_i^\dagger L_i,$$

we get

$$\dot{\rho} = \frac{1}{2}\sum_i \left([L_i\rho, L_i^\dagger] + [L_i, \rho L_i^\dagger]\right). \tag{D.12}$$

Consider now one scattering channel with the appropriate Lindblad operator L, in the same general form as (D.10),

$$L = a_0\mathbf{1} + \vec{a}\;\vec{\sigma}. \tag{D.13}$$

The time evolution of ρ is now given by the corresponding commutator relations for Pauli matrices

$$[\sigma_i, \sigma_j] = 2\;i\;\varepsilon_{ijk}\sigma_k. \tag{D.14}$$

The next calculations refer to the evaluation of the commutators appearing in (D.12).

$$\begin{aligned}[L\rho, L^\dagger] &= [L(u_0\mathbf{1} + \vec{u}\;\vec{\sigma}), L^\dagger] \\ &= u_0[L, L^\dagger] + [L\vec{u}\;\vec{\sigma}, L^\dagger] \\ &= [L, L^\dagger](u_0\mathbf{1} + \vec{u}\;\vec{\sigma}) + L[\vec{u}\;\vec{\sigma}, L^\dagger]\end{aligned} \tag{D.15}$$

$$\begin{aligned}[L, L^\dagger] &= [a_0\mathbf{1} + \vec{a}\;\vec{\sigma}, a_0^*\mathbf{1} + \vec{a}^*\;\vec{\sigma}] \\ &= [\vec{a}\;\vec{\sigma}, \vec{a}^*\;\vec{\sigma}]\end{aligned} \tag{D.16}$$

$$\begin{aligned}[\vec{a}\;\vec{\sigma}, \vec{a}^*\;\vec{\sigma}] &= [\sigma_i, \sigma_j]\;a_i\;a_j^* \\ &= 2\;i\;\varepsilon_{ijk}\;a_i\;a_j^*\;\sigma_k \\ &= 2\;i\;(\vec{a}\times\vec{a}^*)\;\vec{\sigma}.\end{aligned} \tag{D.17}$$

With the decomposition of the complex vector $\vec{a}$ into real and imaginary part

$$\vec{a} = \vec{a}' + i\;\vec{a}'' \tag{D.18}$$

we get for $(\vec{a}\times\vec{a}^*)$:

$$(\vec{a}' + i\;\vec{a}'')\times(\vec{a}' - i\;\vec{a}'') = -2\;i\;\vec{a}'\times\vec{a}''. \tag{D.19}$$

Thus, the commutator of the Lindblad operators is

$$[L, L^\dagger] = 4\ (\vec{a}' \times \vec{a}'')\ \vec{\sigma}, \tag{D.20}$$

and the first term on the right-hand side (rhs) of Eq. (D.15) is

$$[L, L^\dagger](u_0 \mathbf{1} + \vec{u}\ \vec{\sigma}) = 4\ (\vec{a}' \times \vec{a}'')\ \vec{\sigma}\ (u_0 \mathbf{1} + \vec{u}\ \vec{\sigma}), \tag{D.21}$$

with the expression for $(\vec{a}\ \vec{\sigma})\ (\vec{b}\ \vec{\sigma})$:

$$[(\vec{a}' \times \vec{a}'')\ \vec{\sigma}]\ (\vec{u}\vec{\sigma}) = (\vec{a}' \times \vec{a}'')\ \vec{u}\ \mathbf{1} + i\ [(\vec{a}' \times \vec{a}'') \times \vec{u}]\ \vec{\sigma}. \tag{D.22}$$

We evaluate the next commutator

$$\begin{aligned} L[\vec{u}\ \vec{\sigma}, L^\dagger] &= L\ [\vec{u}\ \vec{\sigma}, a_0^* \mathbf{1} + \vec{a}^*\ \vec{\sigma}] \\ &= 2\ i\ L\ (\vec{u} \times \vec{a}^*)\ \vec{\sigma}. \end{aligned} \tag{D.23}$$

With

$$\begin{aligned} L\ (\vec{b}\ \vec{\sigma}) &= (a_0 \mathbf{1} + \vec{a}\ \vec{\sigma})\ (\vec{b}\ \vec{\sigma}) \\ &= a_0\ \vec{b}\ \vec{\sigma} + (\vec{a}\ \vec{\sigma})(\vec{b}\ \vec{\sigma}) \\ &= a_0\ (\vec{b}\ \vec{\sigma}) + (\vec{a}\ \vec{b})\mathbf{1} + i\ (\vec{a} \times \vec{b})\ \vec{\sigma}, \end{aligned} \tag{D.24}$$

and $\vec{b} = \vec{u} \times \vec{a}^*$, Eq. (D.23) gets

$$\begin{aligned} L[\vec{u}\ \vec{\sigma}, L^\dagger] &= 2\ i\ a_0\ (\vec{u} \times \vec{a}^*)\ \vec{\sigma} + 2\ i\ [\vec{a}\ (\vec{u} \times \vec{a}^*)]\ \mathbf{1} \\ &\quad -2\ [\vec{a} \times (\vec{u} \times \vec{a}^*)]\ \vec{\sigma}. \end{aligned} \tag{D.25}$$

Thus, the first term on the rhs of Eq. (D.12) is

$$\begin{aligned} \frac{1}{2}[L\rho, L^\dagger] &= 2\ (\vec{a}' \times \vec{a}'')\ (u_0\ \vec{\sigma}\mathbf{1} + \vec{u}\ \mathbf{1}) + 2\ i\ [(\vec{a}' \times \vec{a}'') \times \vec{u}]\ \vec{\sigma} \\ &\quad +i\ a_0\ (\vec{u} \times \vec{a}^*)\ \vec{\sigma} + i[\vec{a}\ (\vec{u} \times \vec{a}^*)]\ \mathbf{1} \\ &\quad -[\vec{a} \times (\vec{u} \times \vec{a}^*)]\ \vec{\sigma}. \end{aligned} \tag{D.26}$$

The second commutator in Eq. (D.12) can be written as

$$[L, \rho L^\dagger] = \rho\ [L, L^\dagger] + [L, \rho]\ L^\dagger, \tag{D.27}$$

with

$$\begin{aligned} (u_0 \mathbf{1} + \vec{u}\ \vec{\sigma})[L, L^\dagger] &= (u_0 \mathbf{1} + \vec{u}\ \vec{\sigma}) 4 (\vec{a}' \times \vec{a}'')\ \vec{\sigma} \\ &= 4(\vec{a}' \times \vec{a}'')\ (u_0 \vec{\sigma}\mathbf{1} + \vec{u}\ \mathbf{1}) \\ &\quad +4i[\vec{u} \times (\vec{a}' \times \vec{a}'')]\ \vec{\sigma}, \end{aligned} \tag{D.28}$$

and

$$\begin{aligned} [L, \vec{u}\ \vec{\sigma}]\ L^\dagger &= [\vec{a}\ \vec{\sigma}, \vec{u}\ \vec{\sigma}]\ L^\dagger \\ &= 2i\ (\vec{a}\times\vec{u})\ \vec{\sigma}\ L^\dagger \\ &= 2i\ a_0^*\ (\vec{a}\times\vec{u})\ \vec{\sigma} + 2i\ [\vec{a}^*\ (\vec{a}\times\vec{u})]\ \mathbf{1} \\ &\quad -2\ [(\vec{a}\times\vec{u})\times\vec{a}^*]\ \vec{\sigma}. \end{aligned} \tag{D.29}$$

To obtain this result we used the relation

$$\begin{aligned} (\vec{b}\ \vec{\sigma})\ L^\dagger &= (\vec{b}\ \vec{\sigma})\ (a_0^*\ \mathbf{1} + \vec{a}^*\ \vec{\sigma}) \\ &= a_0^*\ (\vec{b}\ \vec{\sigma}) + (\vec{b}\ \vec{\sigma})(\vec{a}^*\ \vec{\sigma}) \\ &= a_0^*\ (\vec{b}\ \vec{\sigma}) + (\vec{a}^*\ \vec{b})\ \mathbf{1} + i(\vec{b}\times\vec{a}^*)\ \vec{\sigma}. \end{aligned} \tag{D.30}$$

Finally[1],

$$\begin{aligned} \frac{1}{2}\ [L, \rho L^\dagger] &= 2\ (\vec{a}'\times\vec{a}'')(u_0\ \vec{\sigma}\mathbf{1} + \vec{u}\ \mathbf{1}) - 2i\ [(\vec{a}'\times\vec{a}'')\times\vec{u}]\ \vec{\sigma} \\ &\quad -i\ a_0^*\ (\vec{u}\times\vec{a})\ \vec{\sigma} - i\ [\vec{a}^*\ (\vec{u}\times\vec{a})]\ \mathbf{1} \\ &\quad -[\vec{a}^*\times(\vec{u}\times\vec{a})]\ \vec{\sigma}. \end{aligned} \tag{D.31}$$

Thus the master equation (D.12) becomes

$$\begin{aligned} \dot{\rho} &= 4(\vec{a}'\times\vec{a}'')\ u_0\ \vec{\sigma} + i\ [a_0\ (\vec{u}\times\vec{a}^*) - a_0^*\ (\vec{u}\times\vec{a})]\ \vec{\sigma} \\ &\quad -[\vec{a}\times(\vec{u}\times\vec{a}^*) + \vec{a}^*\times(\vec{u}\times\vec{a})]\ \vec{\sigma}. \end{aligned} \tag{D.32}$$

With

$$\begin{aligned} a_0\ \vec{a}^* - a_0^*\ \vec{a} &= (a_0' + i a_0'')(\vec{a}' - i\vec{a}'') - (a_0' - i a_0'')(\vec{a}' + i\vec{a}'') \\ &= -2i(a_0'\vec{a}'' - a_0''\vec{a}'), \end{aligned} \tag{D.33}$$

and for $\vec{a}\times(\vec{u}\times\vec{a}^*)$:[2]

$$\begin{aligned} \varepsilon_{ijk}\ a_j\ \varepsilon_{klm}\ u_l\ a_m^* &= \varepsilon_{kij}\ \varepsilon_{klm}\ a_j\ u_l\ a_m^* \\ &= |\vec{a}|^2\ \vec{u} - (\vec{a}\ \vec{u})\ \vec{a}^*, \end{aligned} \tag{D.34}$$

and

$$\begin{aligned} \vec{a}\times(\vec{u}\times\vec{a}^*) + \vec{a}^*\times(\vec{u}\times\vec{a}) &= 2|\vec{a}|^2\ \vec{u} - (\vec{a}\ \vec{u})\ \vec{a}^* - (\vec{a}^*\ \vec{u})\ \vec{a} \\ &= 2|\vec{a}|^2\ \vec{u} - [(\vec{a}' + i\vec{a}'')\ \vec{u}](\vec{a}' - i\vec{a}'') \\ &\quad -[(\vec{a}' - i\vec{a}'')\ \vec{u}](\vec{a}' + i\vec{a}'') \\ &= 2[|\vec{a}|^2\ \vec{u} - (\vec{a}'\ \vec{u})\ \vec{a}' - (\vec{a}''\ \vec{u})\ \vec{a}''], \end{aligned} \tag{D.35}$$

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[1] $\vec{a}\ (\vec{u}\times\vec{a}^*) = \vec{u}\ (\vec{a}^*\times\vec{a}) = 2i\ (\vec{a}'\times\vec{a}'')\ \vec{u}$.

[2] $\varepsilon_{kij}\ \varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

we get for (D.32)

$$\begin{aligned} \dot{\rho} &= 2\big\{2u_0(\vec{a}' \times \vec{a}'') - (a_0'\ \vec{a}'' - a_0''\ \vec{a}') \times \vec{u} \\ &\quad -[|\vec{a}|^2\ \vec{u} - (\vec{a}'\ \vec{u})\ \vec{a}' - (\vec{a}''\ \vec{u})\ \vec{a}'']\big\}\ \vec{\sigma}. \end{aligned} \tag{D.36}$$

This is the master equation [derived from Eq. (D.12) with the help of Eqs. (D.10) and (D.13)] which we use in (4.78).

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