Kepler: From the planets to dark matter

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1 Historical remarks

Johannes Kepler (1571-1630) described the motion of the solar system planets by his famous three laws. From 1594-1600 he was in Graz and worked as a *Landschaftsmathematiker* and became famous because he published a calendar where by chance he made correct weather predictions. From March 1600 to October 1601 he worked in Prague with the famous Tycho Brahe and his task was to calculate the orbit of planet Mars. He realized that the orbit of Mars cannot be a perfect circle but rather an ellipse. Thus he established his first law. In this paper we will however mainly concentrate on his third law which was discovered in 1618 and published 1619 in *Harmonice mundi* (see Fig. 1).

The main idea of Kepler was to describe the planetary system and, as he thought, the whole universe as an harmonic system. There should be perfect harmony and he expressed the orbits of the planets by Platonic solids in his *Mysterium Cosmographicum* that appeared 1596 in Graz. Kepler proposed that the distance relationships between the six planets known at that time could be understood in terms of the five Platonic solids, enclosed within a sphere that represented the orbit of Saturn (Fig. 1).

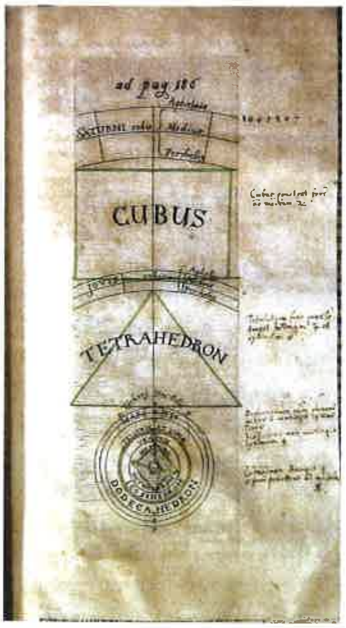

Fig. 1: Kepler used the Platonic solids to determine the position of the planets in his "Harmonice mundi".

2 Kepler's third law

2.1 A simple derivation

Here we give a very simple derivation for Kepler's third law by just considering circular orbits. We assume that the centrifugal force acting on a planet because of its circular motion around the Sun is balanced by the gravitational attraction. m_s is the mass of the Sun, m_P the mass of the planet, r the distance of the planet from the Sun and ω the angular velocity:

$$m_p r\omega^2 = G\frac{m_s m_P}{r^2} \qquad (1)$$

$\omega = 2\pi/T$, T the orbital period. We obtain for orbit radius $r = a$ Kepler's third law:

$$\frac{a^3}{T^2} = \frac{1}{4\pi^2} \cdot G \cdot m_\odot = 3.38 \times 10^{18}\ \mathrm{m^3/s^2}$$

using the gravitational constant $G = 6.67 \times 10^{-11}\ \mathrm{m^3/(kg \cdot s^2)}$. Expressing a in AU (1 AU = 150 Mio km) and T in years yields

$$\frac{a^3}{T^2} = const \qquad (2)$$

This holds for any planetary system. However, since the value of the constant depends on the mass of the central star, it is different for every system. In Table 1 we give as example the values for the constant for some objects in the solar system and in the recently discovered Trappist system, which is an exoplanet system about 40 light years away from us.

Planet	a [AU]	T	const
Mercury	0.395	0.241 y	1.002
Venus	0.723	0.615 y	1.001
Earth	1.000	1.000 y	1.000
Mars	1.574	1.881 y	1.000
Neptune	30.07	161.7 y	0.996
Trappist 1b	0.0115	1.51 d	0.0888
Trappist 1e	0.0293	6.10 d	0.0900
Trappist 1 g	0.0469	12.36 d	0.0899
Trappist 1 h	0.0619	18.76 d	0.0897

Table 1: Comparison of the constant in Kepler's third law for planets in the solar system and in the Trappist system. The values of T are given in years for solar system planets and in earth days for the Trappist planets.

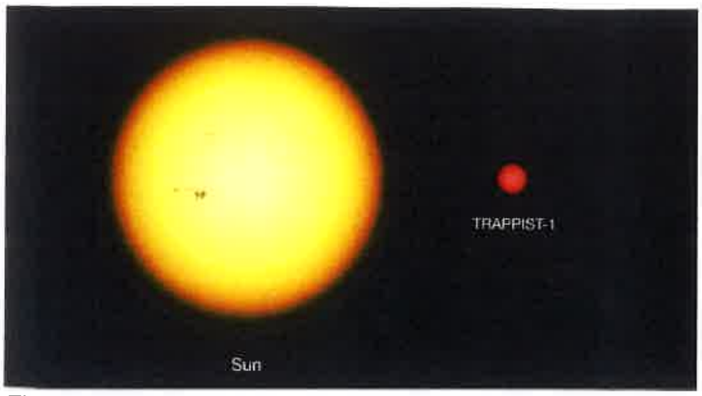

Figure 2: Comparison of the Sun and the star Trappist 1. Source: ESO

The host star has only about 0.089 solar masses. A comparison between the size of the Sun and Trappist 1 is shown in Fig. 2.

The larger differences for the Trappist system result from uncertainties of the measured parameters. They could also be explained due to the stronger gravitational interactions between the relatively close planets.

2.2 Kepler's third law to determine distances

One straightforward application of Kepler's third law is its application to determine distances. Consider two planets 1, 2 then:

$$\frac{a_1^3}{T_1^2} = \frac{a_2^3}{T_2^2} \qquad (3)$$

If the orbit radius in the major axis (which is approximately the distance to the host star (Sun) for small eccentricity of the orbit) of planet 1 is known as well as its orbital period T_1 and the orbital period T_2 of planet 2, we obtain a_2. Thus if one distance of a planet to its host star in a planetary system is known, all other distances follow immediately.

2.3 The exact formulation fo Kepler's third law

Consider two masses M_1, M_2, where the semi major axis of the elliptic orbit of mass M_2 is a and its orbital period T, then in the center of gravity system:

$$\frac{a^3}{T^2} = \frac{G}{4\pi^2}(M_1 + M_2) \qquad (4)$$

Let us use this formula to determine the mass of the Sun. The semi major axis of Earth's orbit is a = 1 AU, the orbital period is 1 year ~ 3 × 10^7 s. Then by neglecting $M_2 << M_1$ we obtain the mass of the Sun: $M_\odot$ = 2 × 10^{30} kg.

3 Masses in the universe

3.1 Stellar masses

The stellar mass is the crucial parameter that determines the lifetime of a star and its final evolution. The lifetime of a star can be directly expressed in terms of its mass by:

$$\tau \sim 10^{10}\left(\frac{M_\odot}{M}\right)^{2.5} y \qquad (5)$$

Thus for the Sun the lifetime is about 10^{10} y, for a star with 5 solar masses it is only 180 million years.

Masses can be determined only in the case a star has a companion and the distance of the companion (could be another star or a planet) to the star as well as its orbital period are known. Then we can directly use the exact formulation of Kepler's third law.

Stellar masses determine the ultimate evolution of a star. Stars lose an important fraction of their mass in their lifetime due to stellar winds, or in the final phases, depending on their mass, in explosions like the ones we see in supernovae. This leads to the following:

- Stellar remnants with masses M < 1.44 $M_\odot$ finally evolve into earth-sized compact white dwarfs. 1.44 solar masses is the Chandrasekhar limit. Below this mass the degenerate electrons can provide the pressure against gravity in the final stellar evolution.
- Stellar remnants with masses 1.44 $M_\odot$ < M_* < 2...3 $M_\odot$ end up as neutron stars (where the pressure of the degenerate neutrons resists gravity).
- Stellar remnants with masses > 2...3 $M_\odot$ end up as black holes where the star completely collapses.

3.2 The mass of the Galaxy

The solar system is just one out of several 100 billion stars in the Milky Way, also called the Galaxy. The mass of the Galaxy can be obtained by stellar statistics, e.g. counting stars in selected fields. A more accurate determination of the mass of the Galaxy results again from Kepler's third law. Our galaxy is a spiral galaxy and the Sun is located at a distance of about 8.5 kpc (about 30.000 light years) from the center of the Galaxy. It orbits around the galactic center in about 220 million years. Using Kepler's third law with the values: a = 8.5 kpc = 8.5 × 10^3 × 3.26 × 10^{16} m = 2.77 × 10^{21} m and T = 220 × 10^6 × 3 × 10^7 s = 6.6 × 10^{15} s leads to a mass of 1.765 × 10^{11} $M_\odot$. Of course this is only an estimate since we neglected all the mass of the Galaxy outside the orbit of the Sun.

Galaxies always occur in clusters and by the dynamics of the cluster members we again can estimate masses.

3.3 Supermassive black holes

The center of our galaxy cannot be observed in the visible part of the electromagnetic spectrum because of interstellar absorption. Observations with high resolution telescopes

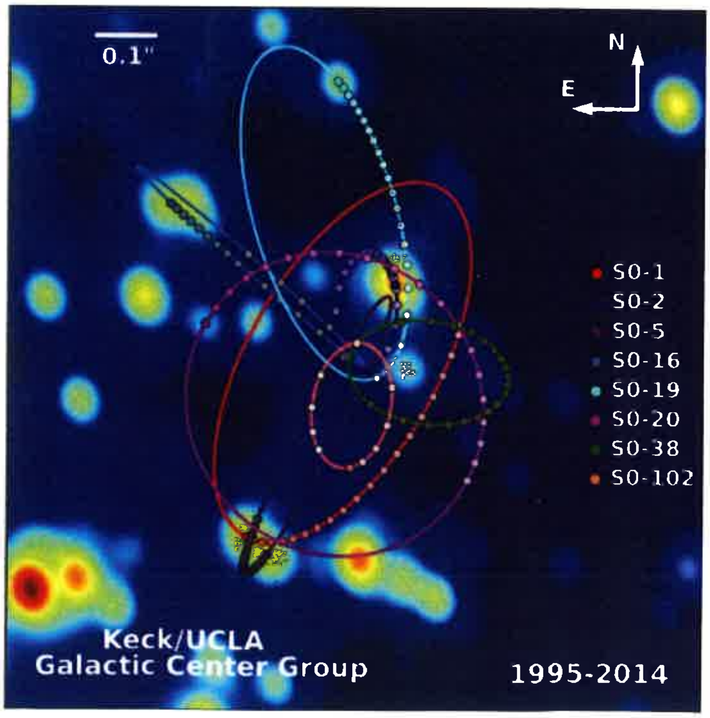

Figure 3: Observations from the Keck telescope; the positions of stars near the galactic center are shown and they indicate a rotation about a central massive object. Source: Keck/UCLA

Figure 4: A galaxy cluster (Abell 1689) that produces gravitational lensing of galaxies behind it. Source: Hubble Space Telescope.

Figure 5: Galaxy cluster (Abell 1689) that produces gravitational lensing of galaxies behind it. Here the distribution of dark matter is indicated by blue color. Source: Hubble Space Telescope.

(like the two 10-m Keck telescopes that can be operated as an interferometer in the IR) show individual stars only several 1000 AU away from the center. Time series of observations covering about 20 years show the motion of such stars about the center (see Fig. 3). Using Kepler's third law we can calculate the central mass around which these stars orbit: for example if $a = 3000$ AU and $T = 40$ years, then the central mass would be 18.6 million solar masses. From the fact that no radiation is received from the central star, it can be concluded that it could be a supermassive black hole. The formation of supermassive black holes, which are observed also in other galaxies' centers, must have occurred during the early evolution of the universe.

4 Dark matter

4.1 First hints for dark matter

Galaxies occur in clusters, for example our Galaxy belongs to the so called local group (a prominent member is our neighboring galaxy, the Andromeda Galaxy). In the 1930s the Swiss astronomer Fritz Zwicky examined galaxy clusters. He realized that these clusters cannot be in dynamical equilibrium, since they are unstable and should dissolve over several hundred million years. Since galaxy clusters are observed also at extremely large distances of several billion light years they must be stable and Zwicky introduced the missing mass concept to explain their stability. This additional mass can, for some reason, not be observed.

4.2 Dark matter in galaxies

When we investigate the rotation curve of a galaxy we would expect the following behavior: the farther an object (e.g. a star) is from the center of its host galaxy, the larger should be its period of revolution because

$$\frac{a^3}{T^2} = const \qquad (6)$$

However, it was found that at larger distances from the galactic center, the speed remains constant or in several cases even increases. This can be only explained by the presence of additional matter that does not radiate, called *Dark Matter*. It can be shown that the amount of dark matter is about five times higher than ordinary visible matter. The presence of dark matter around galaxy clusters can also be inferred from gravitational lensing effects. If, seen from us, there is a galaxy behind a galaxy cluster, then because of the presence of a large mass in the galaxy cluster, light from the more distant galaxy will be bended because of space-time curvature according to general relativity theory. It also became clear that the observable matter of a galaxy cluster is not sufficient to explain the lensing effects.

In Fig. 4 the galaxy cluster Abell 1689 is shown. This cluster is at a distance of about 2.2 billion light years. One clearly sees some lensing effects (curved images of fainter galaxies). The deflection of a light beam is deduced from general relativity (the value here is twice the value from Newtonian physics where light is assumed to consist of massive particles):

$$\theta = \frac{4GM}{rc^2} \qquad (7)$$

where M is the mass (in our case the mass of the galaxy cluster), r the distance of the passing light beam from the mass. In Fig. 5 we show the calculated distribution of dark matter (in blue) around the galaxy cluster. This was obtained by modeling the gravitational lensing.

5 Conclusion

We have shown that Kepler's law can be applied to various astrophysical topics, planets, stellar masses, galaxies and these equations finally even lead us to the concept of dark matter. We give also some simple numerical examples of these applications.

Literature

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