

Convergence to Equilibrium for the Linearized Cometary Flow Equation.

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Abstract

We study convergence to equilibrium for certain spatially inhomogeneous kinetic equations, such as discrete velocity models or a linearization of a kinetic model for cometary flow. For such equations, the convergence to a unique equilibrium state is the result of, firstly, the dissipative effects of the collision operator, which morphs the solution towards an entropy minimizing local equilibrium state, and secondly, the transport operator as well as the imposed periodic boundary conditions, which repulse the solution from the set of local equilibria as long as the approached local equilibrium is not the global one. This behaviour is quantified in a system of differential inequalities of relative entropies with respect to different (sub)classes of local equilibria, respectively, the global equilibrium. We introduce projection operators leading to a convenient notation.

Keywords: inhomogeneous kinetic equation, wave-particle collision operator, cometary flows, convergence to global equilibrium

AMS subject classification: 35Q60, 76P05, 82D10

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1 Introduction

We are interested in the following kinetic transport model called the cometary flow equation:

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(u_f + |v - u_f| \omega) d\omega - f =: Q(f), \quad (1)$$

where $f(t, x, v)$ is a nonnegative particle distribution function depending on time $t > 0$, on position $x \in \mathbb{T}^d$ (the d -dimensional torus with periodic boundary conditions), and on velocity $v \in \mathbb{R}^d$. The collision operator Q is used in quasi-linear plasma theory as a simplified model for wave-particle interaction in cometary flows (see e.g. [7] and the references therein). The first term is a projection (with S^{d-1} and $|S^{d-1}|$ denoting the unit sphere in \mathbb{R}^d and its $(d-1)$ -dimensional Lebesgue measure, respectively) onto the set of distribution functions isotropic around the mean velocity $u_f(t, x)$, which is defined as the fraction of the momentum density $m_f(t, x)$ and the mass density $\rho_f(t, x)$:

$$\rho_f = \int_{\mathbb{R}^d} f dv, \quad m_f = \rho_f u_f = \int_{\mathbb{R}^d} v f dv. \quad (2)$$

Existence and uniqueness of solutions of initial value problems for (1) have been investigated in [7] and in [15], where also the long time behaviour is investigated. A weak convergence result on compact time intervals shifted to infinity is proven similarly to the corresponding result by Desvillettes [9] for the gas dynamics case. By entropy dissipation arguments it is shown that in the limit both the left hand side and the right hand side of (1) vanish.

The set of equilibrium distributions satisfying $Q(f) = 0$ is infinite dimensional. It consists of all velocity distributions which are isotropic around an arbitrary mean velocity. The collision invariants are the components of v as well as all functions of the form $\psi(|v - u_f|)$, i.e.

$$\int_{\mathbb{R}^d} Q(f) v dv = \int_{\mathbb{R}^d} Q(f) \psi(|v - u_f|) dv = 0,$$

for all f . Out of those, only $1, v$, and $|v|^2 = |v - u_f|^2 + 2v \cdot u_f - |u_f|^2$ are independent of f and, thus, produce macroscopic conservation laws. For this reason it is not known how to identify large time limits of solutions of (1) uniquely from the initial data. This in turn prevents the applicability of the entropy dissipation approach for inhomogeneous kinetic equations recently developed by Desvillettes and Villani [10], [12] (see also [14]) which provides strong convergence at algebraic rates as time tends to infinity.

As a consequence, we restrict our attention in this work to a linearized version of (1), which still possesses an infinite dimensional set of equilibrium distributions, but however also possesses enough macroscopic conservation laws such that the limit as $t \rightarrow \infty$ can be uniquely determined from the initial data. For the linearized cometary flow equation, presented in the following section, the Desvillettes-Villani approach is carried out. Our main convergence result is stated in section 2 and proved in sections 3 and 4. In section 3 a system of differential inequalities is derived for a number of relative entropies with respect to certain partial equilibria. In section 4 it is proved that these inequalities imply convergence to equilibrium at arbitrary algebraic rates.

Finally, in section 5, a simple three velocity model is considered which reproduces some of the difficulties found in the linearized cometary flow equation. The entropy dissipation approach can also be carried out with an analogous result. A spectral analysis, however, proves exponential convergence to equilibrium. This example is an extension of the two velocity model considered in [14].

2 The Linearized Cometary Flow Equation

We linearize (1) around an equilibrium steady state of the form $F(|v|^2/2)$, normalized such that $\int_{\mathbb{R}^d} F dv = 1$. Denoting the perturbation by g , the cometary flow equation becomes (see e.g. [6])

$$\partial_t g + v \cdot \nabla_x g = P(g) - g =: LQ(g), \quad (3)$$

with the projection

$$P(g) = \bar{P}(g) - F' v \cdot m_g, \quad (4)$$

and the spherical average

$$\bar{P}(g)(v) = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} g(|v|\omega) d\omega. \quad (5)$$

In (3) LQ denotes the linearized collision operator. It is easily seen that the components of v and all functions of the form $\psi(|v|)$ are collision invariants, i.e.,

$$\int_{\mathbb{R}^d} LQ(g)v dv = \int_{\mathbb{R}^d} LQ(g)\psi(|v|) dv = 0,$$

providing (with $\psi(|v|) = \delta(|v| - |v_0|)$) the global conservation laws

$$\frac{d}{dt} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} v g(t, x, v) dv dx = 0, \quad (6)$$

$$\frac{d}{dt} \int_{\mathbb{T}^d} \int_{S^{d-1}} g(t, x, |v_0| \omega) d\omega dx = 0, \quad (7)$$

for every $|v_0| \geq 0$.

The kernel of the collision operator LQ consists of all velocity distributions of the form $G(|v|^2/2) - F'(|v|^2/2) v \cdot m$ with an arbitrary function G of one variable and an arbitrary vector $m \in \mathbb{R}^d$. Thus, we assume that, as $t \rightarrow \infty$, g converges to an equilibrium distribution

$$g_\infty(x, v) = G_\infty \left(x, \frac{|v|^2}{2} \right) - F' \left(\frac{|v|^2}{2} \right) v \cdot m_\infty(x). \quad (8)$$

It is a consequence of the stationary version of (3) that g_∞ is x -independent:

Lemma 2.1. *Assume that G_∞ and m_∞ are smooth and that g_∞ , given by (8), solves (3) subject to periodic boundary conditions in x . Then G_∞ and m_∞ are independent of x .*

Proof. Substituting (8) into (3) yields

$$v \cdot \nabla_x G_\infty - F' v^{tr} \cdot \nabla_x m_\infty \cdot v = 0. \quad (9)$$

Now we set $v = |v| \omega$ and obtain

$$\omega \cdot \nabla_x G_\infty - F' |v| \omega^{tr} \cdot \nabla_x m_\infty \cdot \omega = 0, \quad \forall \omega \in S^{d-1}, \quad (10)$$

implying that $\nabla_x G_\infty = 0$ holds and that $\nabla_x m_\infty$ is skew-symmetric. Now, a result of Desvillettes [8] implies that $m_\infty(x) = \Lambda x + C$, which can only satisfy periodic boundary conditions iff $\Lambda = 0$. \square

We consider (3) for $t > 0$, $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$, subject to the initial conditions

$$g(0, x, v) = g_I(x, v), \quad (11)$$

where, without loss of generality, we assume vanishing initial total momentum, i.e.

$$\int_{\mathbb{T}^d} \int_{\mathbb{R}^d} v g_I(x, v) dv dx = 0. \quad (12)$$

Then, the conservation of momentum (6) implies vanishing total momentum for all $t > 0$ and, together with the family of conservation laws (7), uniquely determines the global equilibrium g_∞ as

$$g_\infty(v) = G_\infty(|v|^2/2) = \frac{1}{|\mathbb{T}^d|} \int_{\mathbb{T}^d} \bar{P}(g_I)(x, |v|^2/2) dx. \quad (13)$$

However, the smoothness assumption in lemma 2.1 cannot be proven in general although it is necessary: Formally, a distribution $g_\infty(x, v) = \rho(x)\delta(v)$ with an arbitrary x -periodic function $\rho(x)$ is also a stationary solution of (3). Moreover, we conjecture that even for smooth solutions, which are close to a delta distribution centered at the origin in velocity space, convergence to equilibrium can be arbitrarily slow. In order to avoid this problem, we make strong assumptions on the data :

Assumption 2.1. *There exists a lower "cutoff-velocity" $v_0 > 0$ such that*

$$F' \left(\frac{|v|^2}{2} \right) = 0, \quad g_I(x, v) = 0, \quad \text{for } |v| < v_0, \quad (14)$$

and

$$F' \left(\frac{|v|^2}{2} \right) < 0, \quad \text{for } |v| > v_0. \quad (15)$$

Furthermore, $|F'|$ has moments of all orders, i.e. $\int_{\mathbb{R}^d} |v|^k |F'(|v|^2/2)| dv < \infty$, for all $k \geq 0$.

It is an immediate consequence of (14) that $g(t, x, v) = 0$ for $|v| < v_0$, i.e., no perturbation of the nonlinear equilibrium distribution $F(|v|^2/2)$ occurs around $v = 0$.

We remark that assumption (15) is needed for the definition of an entropy: Introducing the measure

$$d\mu = \frac{dx dv}{|F'(|v|^2/2)|}, \quad (16)$$

on the phase space $R = \mathbb{T}^d \times \{v \in \mathbb{R}^d : |v| > v_0\}$, an easy computation shows - provided (15) - the basic entropy inequality

$$\frac{d}{dt} \int_R g^2 d\mu = -2 \int_R (LQ(g))^2 d\mu \leq 0, \quad (17)$$

which is the starting point of our analysis below.

Our main convergence result is proven under assumptions of boundedness and smoothness of solutions, which we are unable to prove. Nevertheless, similar properties have been shown recently for simpler models ([14], [17]).

Assumption 2.2. *The initial value problem (3), (11) has a unique solution satisfying*

$$|g(t, x, v)| \leq C \sqrt{1 + |v|^2} \left| F' \left(\frac{|v|^2}{2} \right) \right|,$$

for $t > 0$ and $(x, v) \in R$, and uniformly in t for all multiindices (k_1, \dots, k_d)

$$\int_R \left(\frac{\partial^{k_1 + \dots + k_d} g}{\partial x_1^{k_1} \dots \partial x_d^{k_d}} \right)^2 d\mu < \infty.$$

Theorem 2.2. *Let the initial data $g_I(x, v)$ satisfy (12) and suppose that the assumptions 2.1 and 2.2 hold. Let g_∞ denote the global equilibrium given by (13). Then, for every $\varepsilon > 0$ there exists $C(\varepsilon, v_0, F) > 0$ such that for all $t > 0$*

$$\int_R (g - g_\infty)^2 d\mu \leq C(\varepsilon, v_0, F) t^{-1/\varepsilon}.$$

3 The Entropy Dissipation Approach

The basic entropy equality (17) suggests to introduce the scalar product

$$\langle f, g \rangle_\mu := \int_R f g d\mu,$$

and the corresponding weighted L^2 -space with the induced norm $\|\cdot\|_\mu$. We also introduce the relative entropy of f with respect to g by

$$H(f|g) := \|f - g\|_\mu^2.$$

In particular, the following entropy dissipation equality is derived analogously to (17) as a consequence of the symmetry of LQ with respect to $\langle \cdot, \cdot \rangle_\mu$:

$$\frac{d}{dt} H(g|g_\infty) = -2H(g|P(g)). \quad (18)$$

In this context we use the terminology 'global equilibrium' for g_∞ and 'local equilibrium' for $P(g)$. Equation (18) already shows the basic difficulty of the entropy dissipation approach for inhomogenous kinetic equations: The decay of the entropy tends to stop, whenever the solution is approaching local equilibrium even without having reached the global equilibrium yet. The central idea of the method introduced in [10], [12] is to quantify how g cannot stay close to a local equilibrium as long as this is not the unique global equilibrium.

This was done in [10], [14], [18] for models with a single conservation law by deriving a second order differential inequality for $H(g|P(g))$ of the form

$$\frac{d^2}{dt^2} H(g|P(g)) \geq \kappa H(g|g_\infty) - C(\varepsilon) H(g|P(g))^{1-\varepsilon}, \quad (19)$$

with positive constants κ and $C(\varepsilon)$. Note that, whenever g is sufficiently close to $P(g)$ in relative entropy, (19) implies convexity in time and $H(g|P(g))$ will return to dissipate entropy in (18) as long as global equilibrium is not reached.

In the present situation, as for the Boltzmann equation [12], such an inequality does not hold, since (see below) an intermediate equilibrium between $P(g)$ and g_∞ has to be quantified as well.

However, we start by calculating the second order time derivative of the relative entropy with respect to the local equilibrium

$$\begin{aligned} \frac{d^2}{dt^2} H(g|P(g)) &= -2\langle LQ(v \cdot \nabla_x g), v \cdot \nabla_x g \rangle_\mu + 4H(g|P(g)) \\ &\quad -6\langle LQ(g), v \cdot \nabla_x g \rangle_\mu + 2\langle LQ(\nabla_x g) \cdot v, v \cdot \nabla_x g \rangle_\mu. \end{aligned} \quad (20)$$

Note that if g is in local equilibrium, i.e. when we set $g = P(g)$ in the right hand side of (20), then all the terms vanish, except for the first, which we rewrite as

$$\begin{aligned} &-\langle LQ(v \cdot \nabla_x g), v \cdot \nabla_x g \rangle_\mu = \|\nabla_x \cdot LQ(vg)\|_\mu^2 \\ &= \|\nabla_x \cdot LQ(vP(g))\|_\mu^2 + \langle \nabla_x \cdot LQ(v(g - P(g))), \nabla_x \cdot LQ(v(g + P(g))) \rangle_\mu. \end{aligned} \quad (21)$$

Considering the first term in the right-hand-side of (21), we denote the energy $e_g(t, x) = \int_{\mathbb{R}^d} |v|^2 \bar{P}(g) dv = \int_{\mathbb{R}^d} |v|^2 g dv$ and recall (4) to derive the following identities :

$$\begin{aligned} P(vP(g)) &= -\frac{|v|^2}{d} m_g F' - \frac{v}{d} F' e_g, \\ LQ(vP(g)) &= \left(v \otimes v - \frac{|v|^2}{d} \right) m_g F' - v \bar{P}(g) - \frac{v}{d} F' e_g, \\ \nabla_x \cdot LQ(vP(g)) &= \left(v \otimes v - \frac{|v|^2}{d} \right) : A F' - \nabla_x \cdot \left(v \bar{P}(g) + \frac{v}{d} F' e_g \right), \end{aligned} \quad (22)$$

where

$$A = \{\nabla_x m_g\} = \frac{1}{2}(\nabla_x m_g + \nabla_x m_g^{tr}) - \frac{1}{d}(\nabla_x \cdot m_g) I_d.$$

Hence, since the two terms of the last identity in (22) are orthogonal with respect to $\langle \cdot, \cdot \rangle_\mu$,

$$\|\nabla_x \cdot LQ(vP(g))\|_\mu^2 = \left\| \left(v \otimes v - \frac{|v|^2}{d} \right) : A F' \right\|_\mu^2 + \left\| v \cdot \nabla_x \left(\bar{P}(g) + \frac{e_g}{d} F' \right) \right\|_\mu^2. \quad (23)$$

For the first term on the right-hand side of (23), we use $I_{ijk} = \int_{\mathbb{R}^d} v_i v_j v_k^2 |F'| dv$

and $I_{ijk} = 0$ for $i \neq j$, $I_{iik} = \frac{e_F}{d}$ for $i \neq k$, $I_{kkk} = \frac{3e_F}{d}$, where $e_F = \int_{\mathbb{R}^d} |v|^2 F dv$:

$$\begin{aligned} \left\| \left(v \otimes v - \frac{|v|^2}{d} \right) : AF' \right\|_{\mu}^2 &= \frac{e_F}{d} \int_{\mathbb{T}^d} \left[3 \sum_i A_{ii}^2 + 2 \sum_{i < j} A_{ii} A_{jj} + 2 \sum_{i < j} A_{ij}^2 \right] dx \\ &= \frac{e_F}{d} \int_{\mathbb{T}^d} \left[\sum_{i,j} A_{ij}^2 + 2 \sum_i A_{ii}^2 + \sum_{i \neq j} A_{ii} A_{jj} \right] dx \\ &= \frac{e_F}{d} \int_{\mathbb{T}^d} \left[\sum_{i,j} A_{ij}^2 + 2 \sum_i A_{ii}^2 - \sum_i A_{ii}^2 \right] dx \\ &\geq \frac{e_F}{d} \int_{\mathbb{T}^d} |A|^2 dx. \end{aligned}$$

Collecting these estimates, we have

$$\frac{d^2}{dt^2} H(g|P(g)) \Big|_{g=P(g)} \geq \frac{2e_F}{d} \int_{\mathbb{T}^d} |\{\nabla_x m_g\}|^2 dx + 2 \left\| v \cdot \nabla_x \left(\bar{P}(g) + \frac{e_g}{d} F' \right) \right\|_{\mu}^2. \quad (24)$$

The first term can be estimated from below by $\frac{2e_F}{d} \int_{\mathbb{T}^d} |\nabla_x m_g| dx$ using a Korn inequality (see [12, proposition 11]), which shows that this term only vanishes for x -independent m_g . The second term, instead of controlling $\nabla_x \bar{P}(g)$, contains the projection

$$P_0(g) = -\frac{e_g}{d} F' = \frac{e_g}{d} |F'|, \quad (25)$$

and therefore vanishes whenever $(I - P_0)(\bar{P}(g))$ is x -independent, which allows still an x -dependent contribution $P_0(\bar{P}(g))$ and (24) is not sufficient to conclude convergence to the equilibrium g_{∞} (13). A similar difficulty occurs also for the Boltzmann equation in [12], which motivates the following procedure.

Our strategy is to decompose $P(g)$ as

$$P(g) = P_0(g) + P_1(g), \quad (26)$$

and then to introduce an intermediate (between local and global) equilibrium, defined as

$$\tilde{P}(g) = P_0(g) + P_1(g_{\infty}), \quad (27)$$

which can alternatively be written as

$$\tilde{P}(g) = P_0(g) + P(g_{\infty}) - P_0(g_{\infty}) = g_{\infty} + P_0(g - g_{\infty}), \quad (28)$$

which will be used below.

Lemma 3.1.

$$H(\tilde{P}(g)|g_\infty) \geq \frac{1}{2}H(g|g_\infty) - H(g|\tilde{P}(g)). \quad (29)$$

Proof. The proof is immediate from the fact that

$$H(\tilde{P}(g)|g_\infty) = H(g|g_\infty) + H(g|\tilde{P}(g)) - 2\langle g - g_\infty, g - \tilde{P}(g) \rangle_\mu.$$

□

We now estimate the second term on the right hand side of (24)

$$\begin{aligned} \|v \cdot \nabla_x (\bar{P} - P_0)(g)\|_\mu^2 &= \sum_{i,j=1}^d \int_R v_i v_j \frac{\partial}{\partial x_i} (\bar{P} - P_0)(g) \frac{\partial}{\partial x_j} (\bar{P} - P_0)(g) d\mu \\ &= \sum_i \int_R v_i^2 \left(\frac{\partial}{\partial x_i} (\bar{P} - P_0)(g) \right)^2 d\mu \\ &= \frac{1}{d} \int_R |v|^2 |\nabla_x (\bar{P} - P_0)(g)|^2 d\mu. \end{aligned} \quad (30)$$

At this point we need assumption 2.1 in order to prevent that (30) vanishes in case of g concentrating around $v = 0$. By the lower bound $|v| \geq v_0$ on the phase space R , we continue to estimate

$$\begin{aligned} \|v \cdot \nabla_x (\bar{P} - P_0)(g)\|_\mu^2 &\geq C \|\nabla_x (\bar{P} - P_0)(g)\|_\mu^2 = C \|\nabla_x (\bar{P} - P_0)(g - g_\infty)\|_\mu^2 \\ &\geq C \|(\bar{P} - P_0)(g - g_\infty)\|_\mu^2, \end{aligned} \quad (31)$$

by a Poincare inequality on \mathbb{T}^d , using that $\int_{\mathbb{T}^d} (\bar{P} - P_0)(g - g_\infty) dx = 0$, point-wise in v . Similarly, $\int_{\mathbb{T}^d} |\nabla_x m_g|^2 dx \geq C \int_{\mathbb{T}^d} |m_g|^2 dx$ holds since $\int_{\mathbb{T}^d} m_g dx = 0$ by the conservation of momentum. Thus, from (24) and (31) it follows with $P_1(g) = (\bar{P} - P_0)(g) - m_g \cdot v F'$ (and these two terms being orthogonal) for a constant κ_1 depending on v_0 and F that

$$\left. \frac{d^2}{dt^2} H(g|P(g)) \right|_{g=P(g)} \geq \kappa_1 \|P_1(g - g_\infty)\|_\mu^2 = \kappa_1 H(g|\tilde{P}(g)), \quad (32)$$

since, for $g = P(g)$, we have by (28) that $g - \tilde{P}(g) = g - g_\infty - P_0(g - g_\infty) = P_1(g - g_\infty)$.

In the following, we apply the same strategy as for (18): first compute the second derivative of the relative entropy with respect to \tilde{P} ,

$$\begin{aligned} \frac{d^2}{dt^2} H(g|\tilde{P}(g)) &= 2\langle (I - P_0)(v \cdot \nabla_x g - LQ(g)), (I - P_0)(v \cdot \nabla_x g) - LQ(g) \rangle_\mu \\ &\quad + 2\langle g - \tilde{P}(g), \nabla_x \cdot (v(v \cdot \nabla_x g) - vLQ(g)) \\ &\quad - LQ(v \cdot \nabla_x g) + LQ(g) + \nabla_x \cdot P_0(-v(v \cdot \nabla_x g) + vLQ(g)) \rangle_\mu \end{aligned} \quad (33)$$

and then consider (33) for $g = \tilde{P}(g)$ with $\tilde{P}(g)(x, v, t) = \frac{e_g(t, x)}{d} F'(|v|^2/2) + P_1(g_\infty)(|v|^2/2)$

$$\begin{aligned}
\frac{d^2}{dt^2} H(g|\tilde{P}(g)) \Big|_{g=\tilde{P}(g)} &= \frac{2}{d^2} \|(I - P_0)(v \cdot \nabla_x e_g F')\|_\mu^2 = \frac{2}{d^2} \|v \cdot \nabla_x e_g F'\|_\mu^2 \\
&= \frac{2}{d^2} \sum_{i,j} \int_R v_i v_j \frac{\partial e_g}{\partial x_i} \frac{\partial e_g}{\partial x_j} |F'| dv dx = \frac{2}{d} \int_{\mathbb{T}^d} |\nabla_x e_g|^2 dx \\
&= \frac{2}{d \int_{\mathbb{R}^d} |F'| dv} \|\nabla_x e_g F'\|_\mu^2 \\
&= \frac{2}{d \int_{\mathbb{R}^d} |F'| dv} \|\nabla_x (\tilde{P}(g) - g_\infty)\|_\mu^2. \tag{34}
\end{aligned}$$

Finally by the Poincare inequality on \mathbb{T}^d , we obtain

$$\frac{d^2}{dt^2} H(g|\tilde{P}(g)) \Big|_{g=\tilde{P}(g)} \geq C H(\tilde{P}(g)|g_\infty). \tag{35}$$

Thus, at least formally, the entropy equation (18) and the inequalities (32) and (35) imply that the decay of $H(g|g_\infty)$ can only stop when global equilibrium is reached. In order to quantify this formal information, we generalize (32) and (35) to all $g \neq P(g)$ and $g \neq \tilde{P}(g)$, respectively. Herein, we will use the following lemma :

Lemma 3.2. *Let Assumption 2.1 be satisfied. Then the operators \bar{P}, P_0, P_1 , and, consequently, P - defined in (5), (25), (26), and (4) - are bounded with respect to $\|\cdot\|_\mu$.*

Proof. The operator $\bar{P}(g)$ is bounded by Jensen's inequality:

$$\|\bar{P}(g)\|_\mu^2 \leq \int_R \bar{P}(g^2) d\mu = \int_R g^2 d\mu = \|g\|_\mu^2. \tag{36}$$

As for the operator P_0 ,

$$\|P_0(g)\|_\mu^2 = \frac{\int_{\mathbb{R}^d} |F'| dv}{d^2} \int_{\mathbb{T}^d} \left(\int_{\mathbb{R}^d} |v|^2 \bar{P}(g) dv \right)^2 dx,$$

we obtain the desired estimate with the Cauchy-Schwartz inequality

$$\begin{aligned}
\|P_0(g)\|_\mu^2 &\leq \frac{\int_{\mathbb{R}^d} |F'| dv}{d^2} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |v|^4 |F'| dv \int_{\mathbb{R}^d} \frac{\bar{P}(g)^2}{|F'|} dv dx \\
&= C \|\bar{P}(g)\|_\mu^2 \leq C \|g\|_\mu^2.
\end{aligned}$$

In order to show that P is bounded, we apply again the Cauchy-Schwartz inequality :

$$\begin{aligned} \|F'v \cdot m_g\|_\mu^2 &\leq \int_R |F'| |v|^2 |m_g|^2 dv dx = C \int_{\mathbb{T}^d} \left(\int_{\mathbb{R}^d} vg dv \right)^2 dx \\ &\leq \int_{\mathbb{R}^d} |v|^2 |F'| dv \int_R \frac{g^2}{|F'|} dv dx = C \|g\|_\mu^2. \end{aligned} \quad (37)$$

Finally, the equations (36) and (37) bound P , and, thus, P_1 . \square

Theorem 3.3. *Let assumptions 2.1 and 2.2 be satisfied. Then,*

$$\frac{d^2}{dt^2} H(g|P(g)) \geq \kappa_1 H(g|\tilde{P}(g)) - \delta H(g|g_\infty) - C_1(\varepsilon) \delta^{\varepsilon-1} H(g|P(g))^{1-\varepsilon} \quad (38)$$

holds for arbitrarily small $1 > \varepsilon > 0$, and $\delta > 0$, and for positive constants κ_1 , and $C_1(\varepsilon)$.

Proof. From (20) und (21),

$$\begin{aligned} \frac{d^2}{dt^2} H(g|P(g)) &= 2 \|\nabla_x \cdot LQ(vP(g))\|_\mu^2 - 6 \langle LQ(g), v \cdot \nabla_x g \rangle \\ &\quad - 2 \langle \nabla_x \cdot LQ(vLQ(g)), \nabla_x \cdot LQ(v(g + P(g))) \rangle_\mu \\ &\quad + 4H(g|P(g)) + 2 \langle v \cdot LQ(\nabla_x g), v \cdot \nabla_x g \rangle_\mu \\ &= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

For the first term, it follows from (32) that $I_1 \geq \kappa_1 H(g|\tilde{P}(g))$.

As for the remaining terms, we begin by estimating I_3 , and for the other integrals similar arguments will apply. For I_3 , the x -independence of g_∞ and integration by parts yields

$$I_3 = 2 \langle \nabla_x (\nabla_x \cdot LQ(vLQ(g))), LQ(v(I + P)(g - g_\infty)) \rangle_\mu. \quad (39)$$

Before we are going to apply Hölder's inequality for (39), we estimate the two factors as (with ∇_x^2 denoting the gradient tensor product)

$$\begin{aligned} |LQ(\nabla_x^2 LQ(g)v)| &\leq |v| \left((I + \bar{P}) (|\nabla_x^2 LQ(g)|) \right. \\ &\quad \left. + \frac{e_g}{d} |F'| \|\nabla_x^2 LQ(g)\|_\mu \right), \end{aligned} \quad (40)$$

$$\begin{aligned} |LQ(v(I + P)(g - g_\infty))| &\leq |v| \left((I + \bar{P})(I + P) (|g - g_\infty|) \right. \\ &\quad \left. + \frac{e_g}{d} |F'| \|g - g_\infty\|_\mu \right). \end{aligned} \quad (41)$$

Note that for the right hand side of (41) assumption 2.2 implies $|g|, |g_\infty| \leq C\sqrt{1+|v|^2}|F'|$, and, thus, $(I+\bar{P})(I+P)(|g-g_\infty|) \leq C\sqrt{1+|v|^2}|F'|$. Therefore, splitting (41) = $(41)^{\varepsilon'}(41)^{1-\varepsilon'}$ for all $1 > \varepsilon' > 0$:

$$\begin{aligned} |I_3| &\leq C \int_R |v|^2 (1+|v|^2)^{\frac{\varepsilon'}{2}} |F'|^{\frac{\varepsilon'}{2}} \\ &\quad \times |F'|^{\frac{\varepsilon'-1}{2}} \left((I+\bar{P})(I+P)(|g-g_\infty|) + \frac{e_g}{d} |F'| \|g-g_\infty\|_\mu \right)^{1-\varepsilon'} \\ &\quad \times |F'|^{-\frac{1}{2}} \left((I+\bar{P})(|\nabla_x^2 LQ(g)|) + \frac{e_g}{d} |F'| \|\nabla_x^2 LQ(g)\|_\mu \right) dv dx \end{aligned}$$

and Hölder's inequality with the exponents $\frac{2}{\varepsilon'}$, $\frac{2}{1-\varepsilon'}$, and 2 yields (with $\int |v|^{\frac{4}{\varepsilon'}} (1+|v|^2) |F'| dv < \infty$ by assumption 2.2) :

$$\begin{aligned} |I_3| &\leq C(\varepsilon') \left\| (I+\bar{P})(I+P)(|g-g_\infty|) + \frac{e_g}{d} |F'| \|g-g_\infty\|_\mu \right\|_\mu^{1-\varepsilon'} \\ &\quad \times \left\| (I+\bar{P})(|\nabla_x^2 LQ(g)|) + \frac{e_g}{d} |F'| \|\nabla_x^2 LQ(g)\|_\mu \right\|_\mu. \end{aligned}$$

Furthermore, by lemma 3.2 and Young's inequality with exponents $\frac{2}{1-\varepsilon'}$ and $\frac{2}{1+\varepsilon'}$

$$\begin{aligned} |I_3| &\leq C(\varepsilon') \|g-g_\infty\|_\mu^{1-\varepsilon'} \|\nabla_x^2 LQ(g)\|_\mu, \\ &\leq \delta H(g|g_\infty) + C(\varepsilon') \delta^{\frac{\varepsilon'-1}{\varepsilon'+1}} \|\nabla_x^2(LQ(g))\|_\mu^{\frac{2}{1+\varepsilon'}}, \end{aligned}$$

for all $\delta > 0$. Finally, the global smoothness assumption 2.2 permits (compare [10]) to control the derivatives of $LQ(g) = P(g) - g$ by the interpolation

$$\|\nabla_x^2 u\|_{L^2(\mathbb{T}^d)} \leq C(\varepsilon') \|u\|_{L^2(\mathbb{T}^d)}^{1-\varepsilon'} \|u\|_{H^n(\mathbb{T}^d)}^{\varepsilon'}, \quad \text{for } n > \frac{2}{\varepsilon'},$$

and with $\frac{1-\varepsilon'}{1+\varepsilon'} = 1 - \varepsilon$:

$$|I_3| \leq \delta H(g|g_\infty) + C(\varepsilon) \delta^{\varepsilon-1} H(g|P(g))^{1-\varepsilon}.$$

In the same manner, we estimate the terms I_2 and I_5 as

$$\begin{aligned} |I_2| &\leq C(\varepsilon') \langle |v| |\nabla_x LQ(g)|, |g-g_\infty| \rangle_\mu, \\ |I_5| &\leq C(\varepsilon') \langle |v|^2 |\nabla_x^2 LQ(g)|, |g-g_\infty| \rangle_\mu, \end{aligned}$$

and we interpolate the derivatives as above for I_3 to match (38).

Finally for I_4 , we note that $H(g|P(g)) \leq CH(g|P(g))^{1-\varepsilon}$ holds by the bounds of assumption 2.2. \square

Theorem 3.4. *Let assumptions 2.1 and 2.2 be satisfied. Then,*

$$\frac{d^2}{dt^2}H(g|\tilde{P}(g)) \geq \kappa_2 H(g|g_\infty) - C_2(\varepsilon) H(g|\tilde{P}(g))^{1-\varepsilon} \quad (42)$$

holds for arbitrarily small $1 > \varepsilon > 0$, and for positive constants κ_2 and $C_2(\varepsilon)$.

Proof. We rewrite (33) with respect to (34) as

$$\begin{aligned} \frac{d^2}{dt^2}H(g|\tilde{P}(g)) &= 2\|(I - P_0)(v \cdot \nabla_x \tilde{P}(g))\|_\mu^2 \\ &\quad + 2\langle v \cdot \nabla_x P_0(g - g_\infty), (I - P_0)(v \cdot \nabla_x(g - \tilde{P}(g))) \rangle_\mu \\ &\quad + 2\langle (I - P_0)(v \cdot \nabla_x(g - \tilde{P}(g))), v \cdot \nabla_x g - \nabla_x \cdot P_0(vg) \rangle_\mu \\ &\quad + 2H(g|P(g)) - 4\langle (I - P_0)(v \cdot \nabla_x g), LQ(g) \rangle_\mu \\ &\quad + 2\langle g - \tilde{P}(g), \nabla_x \cdot (v(v \cdot \nabla_x g)) - \nabla_x \cdot (vLQ(g)) \rangle_\mu \\ &\quad - LQ(v \cdot \nabla_x g) + LQ(g) - \nabla_x \cdot P_0(v(v \cdot \nabla_x g) - vLQ(g)) \rangle_\mu \\ &= \sum_{i=1}^{10} I_i. \end{aligned} \quad (43)$$

and estimate I_1 with (35) and lemma 3.1 as :

$$I_1 \geq C(H(g|g_\infty) - H(g|\tilde{P}(g))).$$

Analogously to the previous proof we estimate I_2 :

$$\begin{aligned} |I_2| &\leq C \int_R |v|^2 |P_0(g - g_\infty)| |\nabla_x^2(g - \tilde{P}(g))| d\mu, \\ &\leq \delta H(g|g_\infty) + C(\varepsilon) \delta^{\varepsilon-1} H(g|\tilde{P}(g))^{1-\varepsilon}. \end{aligned} \quad (44)$$

For I_3 , we apply Hölder's inequality similarly to (41) and (40) after estimating the factors

$$\begin{aligned} |(I - P_0)(\nabla_x^2(g - \tilde{P}(g))v)| &\leq C \left(|v| |\nabla_x^2(g - \tilde{P}(g))| \right. \\ &\quad \left. + |F'| \int |\nabla_x^2(g - \tilde{P}(g))|^2 \frac{dv}{|F'|} \right), \\ |(I - P_0)(v(g - g_\infty))| &\leq C \left(|v| |g - g_\infty| + |F'| \int |g - g_\infty|^2 \frac{dv}{|F'|} \right), \end{aligned}$$

and the second order derivatives are controlled using the same interpolation idea with the global smoothness assumption 2.2 as in the previous proof.

Moreover, $|I_4| \leq H(g|\tilde{P}(g))$ is a consequence of lemma 3.5 below. All the remaining terms I_5 - I_{10} are estimated with similar arguments as in the proof of the previous theorem and yield bounds of the form (44). The proof is completed by choosing δ small enough. \square

Lemma 3.5. *Let assumptions 2.1 and 2.2 be satisfied. Then, the inequalities*

$$H(g|\tilde{P}(g)) - H(g|P(g)) \geq 0, \quad (45)$$

$$\frac{d}{dt} \left(H(g|\tilde{P}(g)) - H(g|P(g)) \right) \leq C(\varepsilon) H(g|g_\infty)^{1-\varepsilon}, \quad (46)$$

hold for arbitrarily small $1 > \varepsilon > 0$ with a positive constant $C(\varepsilon)$.

Proof. The identity $H(g|\tilde{P}(g)) - H(g|P(g)) = \|P_1(g - g_\infty)\|_\mu^2 \geq 0$ proves the first inequality. Differentiation with respect to time gives

$$\frac{d}{dt} \|P_1(g - g_\infty)\|_\mu^2 = 2 \langle P_1(g - g_\infty), P_1(-v \cdot \nabla_x (g - g_\infty) + LQ(g - g_\infty)) \rangle_\mu,$$

which is estimated in the same way as in the previous two proofs. \square

4 A System of Ordinary Differential Inequalities

We introduce $x := H(g|g_\infty)$, $y := H(g|P(g))$, $z := H(g|\tilde{P}(g))$, and $w := z - y$ in (18), (38), (42), (45), and (46), and denote time-derivatives by $\frac{d}{dt} = ' :$

$$x' = -2y, \quad (47)$$

$$y'' \geq \kappa_1 z - \delta x - \delta^{\varepsilon_y - 1} C_1(\varepsilon_y) y^{1-\varepsilon_y}, \quad (48)$$

$$z'' \geq \kappa_2 x - C_2(\varepsilon_z) z^{1-\varepsilon_z}, \quad (49)$$

$$|w'| \leq C_3(\varepsilon_w) x^{1-\varepsilon_w}, \quad (50)$$

where $1 > \varepsilon_y, \varepsilon_z, \varepsilon_w > 0$ and $\delta > 0$ are arbitrarily small, $x, y, z, w \geq 0$, and $\kappa_1, \kappa_2, C_1, C_2$, and C_3 are positive constants.

We want to deduce decay of $x(t)$ with an arbitrarily high algebraic rate according to arbitrarily small $\varepsilon_y, \varepsilon_z, \varepsilon_w > 0$. Note that the first three inequalities could be seen as a 'closed' system for x, y , and z . However, the additional information contained in the fourth inequality shall be needed.

The presented proof is quite particular in quantifying different regimes of (47)–(49) and using (50) to prevent rapid oscillations inbetween. As a preliminary technical result on second-order differential inequalities, we reformulate [12, Lemma 12], which discusses time-averages of the entropy production :

Lemma 4.1. *Let $h \in C^2([0, L])$ be nonnegative and satisfy*

$$h''(t) + C h(t)^{1-\varepsilon} \geq \alpha, \quad \text{for } 0 \leq t \leq L,$$

with positive constants C, α and $\varepsilon \in (0, \frac{1}{10})$. Then,

- either L is small : $L \leq 50 C^{-\frac{1}{2(1-\varepsilon)}} \alpha^{\frac{\varepsilon}{2(1-\varepsilon)}}$,
- or h is large on the average : $\langle h \rangle_{(0,L)} = \frac{1}{L} \int_0^L h(t) dt \geq \frac{1}{100} \left(\frac{\alpha}{C}\right)^{\frac{1}{1-\varepsilon}}$.

Proof. The proof follows from [12, Lemma 12] after the rescaling $\tau = t\sqrt{C}$, $\alpha' = \frac{\alpha}{C}$, $\frac{d^2h}{d\tau^2} + h^{1-\varepsilon} \geq \alpha'$ with $\tau \in [0, L\sqrt{C}]$ and $L\sqrt{C} = L'$. □

Our main result theorem 2.2 is a direct consequence of

Theorem 4.2. *Let x, y, z , and $w = z - y \geq 0$ be smooth and, for $t > 0$, satisfy (47)–(50), where $1 > \varepsilon_y, \varepsilon_z, \varepsilon_w > 0$, and $\delta > 0$ are arbitrarily small. Then, for every sufficiently small $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that*

$$x(t) \leq C(\varepsilon) t^{-1/\varepsilon}. \quad (51)$$

Proof. We view (51) in the following way: let $t_0 > 0$ be arbitrary with $\alpha_0 = x(t_0)$. The aim is to find an upper bound of the form

$$T_0 \leq C\alpha_0^{-\varepsilon} \quad (52)$$

on a time T_0 such that $x(t_0 + T_0) = \gamma\alpha_0$, where $\gamma < 1$ is given.

Once such a bound is proven, (51) follows as in [10], [12].

At first, we consider (48) using $z = w + y \geq w$:

$$y'' \geq \kappa_1 w - \delta x - \delta^{\varepsilon_y - 1} C_1 y^{1-\varepsilon_y}. \quad (53)$$

The idea of the following is to deduce 'big' $\langle y \rangle_{(0,L)}$ -averages from lemma 4.1, where we distinguish between the cases where w is 'big' in (53), and the cases where w is 'small' but y is close to $z = y + w$ and lemma 4.1 is used for (49). However, the realization of this concept requires some care.

Step 1: We define the set Ω_z (quantifying w 'small') by

$$t \in \Omega_z \Leftrightarrow \text{dist}(t, \{t_0 \leq \tau \leq t_0 + T_0 : w(\tau) \geq \tilde{w}(\alpha_0)\}) \geq \mu(\alpha_0), \quad (54)$$

where $\tilde{w}(\alpha_0)$ and $\mu(\alpha_0)$ are to be chosen later. On the interval $[t_0, t_0 + T_0]$, the set Ω_z and its complement divide into unions of intervals : $\Omega_z = \cup I_z$ and $[t_0, t_0 + T_0] \setminus \Omega_z = \cup I_y$ (where w is 'big'), and lemma 4.1 will be applied to (48) and (49) for I_y and I_z , respectively.

Moreover on I_y , we quantify w 'big' using (50), which controls the derivative $|w'|$ in terms of $x \leq x(t_0) = \alpha_0$ (by (47))

$$\begin{aligned} w &\geq \tilde{w}(\alpha_0) - \mu(\alpha_0) \sup_{t_0 < \tau < t_0 + T_0} |w'(\tau)| \\ &\geq \tilde{w}(\alpha_0) - C_3 \alpha_0^{1-\varepsilon_w} \mu(\alpha_0) =: \hat{w}(\alpha_0) > 0. \end{aligned} \quad (55)$$

Step 2: For nonempty intervals I_y , we have by construction of Ω_z (54) that the length $\ell(I_y) \geq \min\{\mu(\alpha_0), T_0\}$. The following three cases are possible :

Case 1) $[t_0, t_0 + T_0] = \Omega_z$ and there are no intervals I_y ,

Case 2) $T_0 \leq \mu(\alpha_0)$ will satisfy (52) for suitable $\mu(\alpha_0)$ to be chosen below,

Case 3) $\ell(I_y) \geq \mu(\alpha_0)$: Firstly, we consider $\delta = \delta(\alpha_0)$ to be fixed below, for which estimate with $\alpha(\alpha_0)$ to be chosen below

$$\kappa_1 w - \delta x \geq \kappa_1 \widehat{w}(\alpha_0) - \delta(\alpha_0)\alpha_0 =: \alpha(\alpha_0) > 0. \quad (56)$$

Then, for (48), lemma 4.1 applies with $C = C_1/\delta(\alpha_0)^{1-\varepsilon_y}$, $\alpha = \alpha(\alpha_0)$, and $L = \ell(I_y)$. Moreover, due to $\ell(I_y) \geq \mu(\alpha_0)$, we rule out the first case in lemma 4.1 by setting

$$\mu(\alpha_0) \geq 50 \left(C_1 \delta(\alpha_0)^{\varepsilon_y - 1} \right)^{-\frac{1}{2(1-\varepsilon_y)}} \alpha(\alpha_0)^{\frac{\varepsilon_y}{2(1-\varepsilon_y)}}. \quad (57)$$

Therefore, the second case of lemma 4.1 yields

$$\langle y \rangle_{I_y} \geq \frac{1}{100} \left(C_1 \delta^{\varepsilon_y - 1} \right)^{-\frac{1}{1-\varepsilon_y}} \alpha(\alpha_0)^{\frac{1}{1-\varepsilon_y}}.$$

Step 3: Next, for the intervals $I_z \subseteq \Omega_z$, it follows by (47) that

$$\kappa_2 x \geq \kappa_2 x(t_0 + T_0) = \kappa_2 \gamma \alpha_0.$$

Then, applying lemma 4.1 to (49) yields

- either : $\ell(I_z) \leq 50 C_2^{-\frac{1}{2(1-\varepsilon_z)}} (\kappa_2 \gamma \alpha_0)^{\frac{\varepsilon_z}{2(1-\varepsilon_z)}}$,
- or : $\langle z \rangle_{I_z} \geq \frac{1}{100} \left(\frac{\kappa_2 \gamma \alpha_0}{C_2} \right)^{\frac{1}{1-\varepsilon_z}}$.

In the second case, equation (54) implies with the constant $a_1 = \frac{1}{100} \left(\frac{\kappa_2 \gamma}{C_2} \right)^{\frac{1}{1-\varepsilon_z}}$

$$\langle y \rangle_{I_z} = \langle z - w \rangle_{I_z} \geq a_1 \alpha_0^{\frac{1}{1-\varepsilon_z}} - \widetilde{w}(\alpha_0) \geq \frac{a_1}{2} \alpha_0^{\frac{1}{1-\varepsilon_z}},$$

where we have chosen $\widetilde{w}(\alpha_0) = \frac{a_1}{2} \alpha_0^{\frac{1}{1-\varepsilon_z}}$. Moreover, we set in the definition of Ω_z (54) and in (56) the choices

$$\begin{aligned} \mu(\alpha_0) &= \frac{a_1}{4C_3} \alpha_0^{\frac{\varepsilon_z}{1-\varepsilon_z} + \varepsilon_w} \implies \widehat{w}(\alpha_0) = \frac{a_1}{4} \alpha_0^{\frac{1}{1-\varepsilon_z}}, \\ \delta(\alpha_0) &= \frac{\kappa_1 a_1}{8} \alpha_0^{\frac{\varepsilon_z}{1-\varepsilon_z}} \implies \alpha(\alpha_0) = \frac{a_1}{8} \kappa_1 \alpha_0^{\frac{1}{1-\varepsilon_z}}. \end{aligned} \quad (58)$$

By inserting (58) into (57) we get the constraint

$$\alpha_0^{\frac{\varepsilon_y}{(1-\varepsilon_y)(1-\varepsilon_z)} - \frac{\varepsilon_z}{1-\varepsilon_z} - 2\varepsilon_w} \leq C_1^{\frac{1}{1-\varepsilon_y}} a_2, \quad (59)$$

where - for small ε_y , ε_z , and ε_w - the constant a_2 can be chosen to depend only on γ , κ_1 , κ_2 , C_2 , and C_3 . In the following, we choose $\varepsilon_y \leq \frac{1}{2}$ and $\varepsilon_z = \varepsilon_w = \frac{\varepsilon_y}{4}$. Thus the exponent on the right-hand-side of (59) is positive, and (59) can be satisfied for all possible values of $\alpha_0 \in [0, x(t=0)]$ by making C_1 bigger if necessary (which does not conflict with (48)).

We summarize step 2 and step 3 that for every I_y

$$\langle y \rangle_{I_y} \geq a_3 \alpha_0^{\frac{\varepsilon_z}{1-\varepsilon_z} + \frac{1}{(1-\varepsilon_y)(1-\varepsilon_z)}}, \quad (60)$$

and for every I_z that either

$$\ell(I_z) \leq 50 \alpha_0^{\frac{\varepsilon_z}{2(1-\varepsilon_z)}} \quad \text{or} \quad \langle y \rangle_{I_z} \geq \frac{a_1}{2} \alpha_0^{\frac{1}{1-\varepsilon_z}}. \quad (61)$$

Step 4: We continue by combining pairs of intervals (I_y, I_z) to their union $\bar{I} := I_y \cup I_z$, where we restrict to the above case 3) : $\ell(I_y) \geq \mu(\alpha_0)$. Then,

$$\langle y \rangle_{\bar{I}} = \frac{\ell(I_y) \langle y \rangle_{I_y} + \ell(I_z) \langle y \rangle_{I_z}}{\ell(\bar{I})} = \frac{\ell(I_y)}{\ell(I_y) + \ell(I_z)} \langle y \rangle_{I_y} + \frac{\ell(I_z)}{\ell(I_y) + \ell(I_z)} \langle y \rangle_{I_z}. \quad (62)$$

and we consider the two cases according to (61) :

1. In the first case in (61), $\ell(I_z) \leq 50 \alpha_0^{\frac{\varepsilon_z}{2(1-\varepsilon_z)}}$ implies

$$\frac{\ell(I_y)}{\ell(I_z)} \geq a_4 \alpha_0^{\frac{\varepsilon_z}{2(1-\varepsilon_z)} + \varepsilon_w},$$

with a constant a_4 (and all constants $a_{j=5,\dots}$ from now on) being independent from α_0 , ε_y , ε_z , and ε_w . Consequently,

$$\frac{\ell(I_y)}{\ell(I_y) + \ell(I_z)} \geq a_5 \alpha_0^{\frac{\varepsilon_z}{2(1-\varepsilon_z)} + \varepsilon_w},$$

and, neglecting the last term in (62), we obtain with an exponent $\varepsilon_1(\varepsilon_y, \varepsilon_z, \varepsilon_w) > 0$ tending to zero as $\varepsilon_y, \varepsilon_z, \varepsilon_w \rightarrow 0$ that

$$\langle y \rangle_{\bar{I}} \geq a_6 \alpha_0^{1+\varepsilon_1}. \quad (63)$$

2. For the second case in (61) and (60), both the mean values on I_y and I_z satisfy already estimates of the form (63).

Step 5: Finally, we regard the complete interval $[t_0, t_0 + T_0]$, where we detail further the cases 1) – 3) from step 2 :

1a) $[t_0, t_0 + T_0] = \Omega_z$ and $T_0 \leq 50 \alpha_0^{\frac{\varepsilon_z}{2(1-\varepsilon_z)}}$.

1b) $[t_0, t_0 + T_0] = \Omega_z$ and

$$\langle y \rangle_{[t_0, t_0 + T_0]} \geq \frac{a_1}{2} \alpha_0^{\frac{1}{2(1-\varepsilon_z)}} = a_7 \alpha_0^{1+\varepsilon_1}. \quad (64)$$

Integration of (47) yields $\alpha_0(1 - \gamma) = 2T_0 \langle y \rangle_{[t_0, t_0 + T_0]}$ and, thus, for an $\varepsilon_2(\varepsilon_y, \varepsilon_z, \varepsilon_w) > 0$ tending to zero as $\varepsilon_y, \varepsilon_z, \varepsilon_w \rightarrow 0$,

$$T_0 \leq a_8 \alpha_0^{-\varepsilon_2}. \quad (65)$$

2) $T_0 \leq \mu(\alpha_0)$ immediately implies an estimate of the form (65).

3a) $\ell(I_y) \geq \mu(\alpha_0)$ for all I_y , and $\#I_y \geq \#I_z$, (where $\#I_y$ and $\#I_z$ denote the numbers of I_y and, respectively, I_z). We can split $[t_0, t_0 + T_0]$ into intervals $I = \bar{I} = I_y \cup I_z$ or $I = I_y$, where

$$\langle y \rangle_I \geq a_9 \alpha_0^{1+\varepsilon_1},$$

holds by (60) and (63), which further implies (64) and, thus, (65).

3b) $\ell(I_y) \geq \mu(\alpha_0)$ for all I_y and $\#I_z = \#I_y + 1$. According to the two cases in (61) for the one extra I_z , we either have the situation of case 3a), or

$$\ell(I_z) \leq 50 \alpha_0^{\frac{\varepsilon_z}{2(1-\varepsilon_z)}} \implies \frac{T_0 - \ell(I_z)}{T_0} \geq a_{10} \alpha_0^{\varepsilon_1},$$

since $T_0 - \ell(I_z) \geq \mu(\alpha_0)$. By splitting $[t_0, t_0 + T_0] = ([t_0, t_0 + T_0] \setminus I_z) \cup I_z$ as in (62) we again obtain (64).

Thus, all cases lead to estimates of the form (65), which completes the proof. \square

5 A Discrete Velocity Model

In this section, we introduce a one-dimensional linear discrete velocity model, for which the entropy dissipation approach leads to the same system of ordinary differential inequalities as for the linearized cometary flow equation. However, the discrete velocity model can be solved explicitly by Fourier expansion, which proves actually exponential convergence to equilibrium. It is interesting to compare the three-velocity model below to the two-velocity model discussed in [14], in which the entropy dissipation approach controls local equilibria already by one second order differential inequality like (19).

We consider the equation

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = Lf, \quad (66)$$

with

$$f = (f_+, f_0, f_-), \quad (67)$$

periodic boundary conditions in $x \in [0, 1)$, and initial condition $f(t = 0) = f_I$. We use a matrix-vector notation, and collect the discrete velocities 1, 0 and -1 in the diagonal matrix

$$v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (68)$$

For the collision operator L , we choose

$$L = \begin{pmatrix} -1/6 & 1/3 & -1/6 \\ 1/3 & -2/3 & 1/3 \\ -1/6 & 1/3 & -1/6 \end{pmatrix}, \quad (69)$$

which can be written as $L = P_0 + P_1 - I = \psi_0 \otimes \psi_0 + \psi_1 \otimes \psi_1 - I$, with

$$\begin{aligned} \psi_0 &= \frac{1}{\sqrt{3}}(1, 1, 1), & P_0 &= \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \\ \psi_1 &= \frac{1}{\sqrt{2}}(1, 0, -1), & P_1 &= \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Pointing out the similarities between (66) and the linearized cometary flow equation, we define - in analogy to section 3 - the entropy for (66)

$$H(f|g) := \|f - g\|^2,$$

where $\|\cdot\|$ denotes the norm induced by the scalar product

$$\langle f, g \rangle := \int_0^1 f \cdot g \, dx.$$

Note that ψ_0 and ψ_1 are collision invariants since

$$\psi_i^{tr} L = 0 \Leftrightarrow L\psi_i = 0.$$

Multiplying (66) by ψ_i , $i = 0, 1$ yields the conservation laws

$$\frac{\partial}{\partial t}(\psi_i \cdot f) + \frac{\partial}{\partial x}(\psi_i^{tr} v f) = 0,$$

where $i = 0$ corresponds to the conservation of mass, and $i = 1$ to the conservation of momentum.

The global equilibrium f_∞ is given by

$$f_\infty = \langle f_I, \psi_0 \rangle \psi_0 + \langle f_I, \psi_1 \rangle \psi_1.$$

The local equilibrium is denoted by Pf , where $Pf = P_0f + P_1f$.

The time-derivative of the relative entropies with respect to the global equilibrium

$$\frac{d}{dt}H(f|f_\infty) = -2H(f|Pf),$$

leads, as in section 3, to consider the second time-derivatives of the relative entropies with respect to the local equilibrium

$$\begin{aligned} \frac{d^2}{dt^2}H(f|Pf) &= -2 \left\langle Lv \frac{\partial f}{\partial x}, v \frac{\partial f}{\partial x} \right\rangle + 4H(f|Pf) \\ &\quad - 6 \left\langle Lf, v \frac{\partial f}{\partial x} \right\rangle + 2 \left\langle vL \frac{\partial f}{\partial x}, v \frac{\partial f}{\partial x} \right\rangle, \end{aligned} \quad (70)$$

which has the same structure as (20) for cometary flow equation.

If we assume that f is in local equilibrium (i.e., $f = Pf$), only the first term on the right hand side of (70) contributes, since $v(-L)v = \frac{1}{3}P_1$:

$$\frac{d^2}{dt^2}H(f|Pf)|_{f=Pf} = \frac{2}{3} \left\| \frac{\partial}{\partial x} P_1 f \right\|^2.$$

However, as for the linearized cometary flow equation, this term may vanish without $f = f_\infty$, and we introduce the projection $\tilde{P}f = P_0f + P_1f_\infty$, with the matrix representation $\tilde{P} = P_0 + P_1 + P_1P_0$. Note that $(P - \tilde{P})f = P_1(f - f_\infty)$, whence

$$\frac{d^2}{dt^2}H(f|\tilde{P}f) = \frac{d^2}{dt^2}H(f|Pf) + 2\left\|P_1v\frac{\partial f}{\partial x}\right\|^2 - 2\left\langle P_1\frac{\partial f}{\partial x}, v^2\frac{\partial f}{\partial x}\right\rangle. \quad (71)$$

By setting $f = \tilde{P}f$ in (71), it immediately follows that

$$\frac{d^2}{dt^2}H(f|\tilde{P}f)|_{f=\tilde{P}f} = \int_0^1 \left(\frac{\partial}{\partial x}(f_+ + f_-)\right)^2 dx = \frac{4}{3}\left\|\frac{\partial}{\partial x}P_0f\right\|^2 \geq CH(f|f_\infty), \quad (72)$$

which vanishes if and only if f is in global equilibrium. Now, arbitrarily fast algebraic convergence to equilibrium follows from theorems analog to 3.3 and 3.4 as well as lemma 3.5, which can be proven analog to section 4.

On the other hand, exponential convergence is shown directly by Fourier expansion,

$$f(x, t) = \sum_{k=-\infty}^{\infty} c_k(t)e^{i2\pi kx}. \quad (73)$$

Substituting (73) into (66), the coefficients compare to

$$\partial_t c_k = (L - i2\pi kv)c_k,$$

and it follows from the definition of L and v that

$$L - i2\pi kv = \begin{pmatrix} -1/6 - i2\pi k & 1/3 & -1/6 \\ 1/3 & -2/3 & 1/3 \\ -1/6 & 1/3 & -1/6 + i2\pi k \end{pmatrix}. \quad (74)$$

The characteristic polynomial of (74) is given by

$$p_k(\lambda) = \lambda^3 + \lambda^2 + 4\pi^2k^2(\lambda + 2/3).$$

For $k = 0$ (and, thus, $\mu_k = 0$) we recover the double zero eigenvalue corresponding to the two dimensional set of equilibrium distributions. The third eigenvalue for $k = 0$ is $\lambda = -1$. For $k \neq 0$, an application of the Routh-Hurwitz criterion shows that all remaining eigenvalues have negative real parts. It is easily shown that, as $|k| \rightarrow \infty$, the three zeroes of p_k are approximated by

$$\lambda_{k1} \approx -\frac{2}{3}, \quad \lambda_{k2} \approx -\frac{1}{6} + 2\pi ki, \quad \lambda_{k3} \approx -\frac{1}{6} - 2\pi ki.$$

This proves the existence of a spectral gap and, thus, exponential convergence to equilibrium for (66).

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