

Quantum field theory

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1 Classical scalar fields

Classical field theory. The action of a system described by classical mechanics is given by

$$S = \int_{t_1}^{t_2} dt L(q_i(t), \dot{q}_i(t)) = \int dt \left(\frac{1}{2} \sum_i \dot{q}_i^2 - V(q_1 \dots q_n) \right). \quad (1.1)$$

The transition to classical field theory proceeds via the replacements

$$q_i(t) \rightarrow \Phi(\mathbf{x}, t) \rightarrow \Phi(x), \quad \dot{q}_i(t) \rightarrow \frac{\partial \Phi(\mathbf{x}, t)}{\partial t} \rightarrow \partial_\mu \Phi(x), \quad (1.2)$$

because in a relativistic theory the time derivative can only appear as a part of ∂_μ . The action then takes the form

$$S = \int d^4x L(\Phi(x), \partial_\mu \Phi(x)) = \int_V d^4x \mathcal{L}(\Phi(x), \partial_\mu \Phi(x)), \quad (1.3)$$

where \mathcal{L} is called the Lagrangian density or simply the **Lagrangian** of the theory.

To obtain the equations of motion, we vary the action with respect to Φ and $\partial_\mu \Phi$ in a given volume V with the boundary condition $\{\delta\Phi, \delta(\partial_\mu \Phi)\}|_{\partial V} = 0$. Hamilton's principle of stationary action $\delta S = 0$ then entails

$$\begin{aligned} 0 \stackrel{!}{=} \delta S &= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \Phi} \delta\Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \delta(\partial_\mu \Phi) \right] \\ &= \int d^4x \left[\left(\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) \delta\Phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \delta\Phi \right) \right], \end{aligned} \quad (1.4)$$

where we interchanged the variation with the derivative and performed a partial integration. The second bracket is a total derivative and can be converted to a surface integral via Gauss' law. It is zero because the field and its derivative vanish at the boundary:

$$\int_V d^4x \partial_\mu F^\mu = \int_{\partial V} d\sigma_\mu F^\mu = 0. \quad (1.5)$$

The remaining integrand must also vanish because $\delta\Phi$ is an arbitrary variation. This leads to the **Euler-Lagrange equations of motion**:

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = 0. \quad (1.6)$$

If the Lagrangian contains several fields $\Phi_i(x)$, one simply has to sum over them in Eq. (1.4) and the equations of motion hold for each component separately.

Finally, let's generalize the Hamiltonian formalism to the field-theoretical description. For a discrete system, the canonical conjugate momentum and Hamilton function are given by

$$p_i(t) = \frac{\partial L}{\partial \dot{q}_i(t)}, \quad H = \sum_i \dot{q}_i p_i - L. \quad (1.7)$$

In the continuum limit, the conjugate momentum becomes the canonically conjugate momentum *density* $\Pi(x)$,

$$p(\mathbf{x}, t) = \frac{\partial L}{\partial \dot{\Phi}(\mathbf{x}, t)} \quad \rightarrow \quad \Pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}(x)}, \quad (1.8)$$

and the Hamilton function acquires the form

$$H = \int d^3x \left(\Pi(x) \dot{\Phi}(x) - \mathcal{L} \right) =: \int d^3x \mathcal{H}(x), \quad (1.9)$$

where $\mathcal{H}(x)$ is the Hamiltonian density.

Real scalar field and Klein-Gordon equation. We start with the simplest example of a field theory. It contains only one type of field: a real scalar field $\Phi(x) = \Phi^*(x)$. What are the possible terms that can appear in the Lagrangian? \mathcal{L} must be a Lorentz scalar, so it can only depend on Φ and $\partial_\mu \Phi \partial^\mu \Phi$ (and higher powers of these expressions). The combination $\partial_\mu \partial^\mu \Phi = \square \Phi$ is a total derivative, so it doesn't change the equations of motion. Based on these considerations we write

$$\mathcal{L} = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^2 - V(\Phi^n, \Phi^n (\partial \Phi)^m). \quad (1.10)$$

The first two terms define the Lagrangian for a free scalar field, whereas the potential V contains higher possible interaction terms.¹ The action is $S = \int d^4x \mathcal{L}$, and we can check that the mass dimensions work out correctly:

$$[S] = 0, \quad [d^4x] = -4, \quad [\mathcal{L}] = 4, \quad [\Phi] = 1, \quad [\partial_\mu] = 1, \quad (1.11)$$

and therefore the parameter m has indeed the dimension of a mass. Discarding the interaction terms (which we will always do in this chapter, hence 'free fields'), we can easily work out the Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \Phi} = -m^2 \Phi, \quad \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} = \partial^\mu \Phi, \quad \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} = \partial_\mu \partial^\mu \Phi = \square \Phi, \quad (1.12)$$

and thereby arrive at the **Klein-Gordon equation**:

$$(\square + m^2)\Phi = 0. \quad (1.13)$$

To derive the Hamiltonian density, we have to find the conjugate momentum:

$$\mathcal{L} = \frac{1}{2} \left(\dot{\Phi}^2 - (\nabla \Phi)^2 - m^2 \Phi^2 \right) \quad \Rightarrow \quad \Pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}(x)} = \dot{\Phi}(x), \quad (1.14)$$

and therefore we obtain

$$\mathcal{H} = \Pi \dot{\Phi} - \mathcal{L} = \Pi^2 - \mathcal{L} = \frac{1}{2} \left(\Pi^2 + (\nabla \Phi)^2 + m^2 \Phi^2 \right). \quad (1.15)$$

¹In the quantum field theory, renormalizability will limit their form to Φ^3 and Φ^4 interactions.

The solutions of the Klein-Gordon equation are plane waves $e^{\pm ipx}$ with dispersion relation $p^2 = m^2 \Rightarrow p_0 = \pm\sqrt{\mathbf{p}^2 + m^2} = \pm E_p$, so we can write its general solutions as

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} (a(\mathbf{p}) e^{-ipx} + a^*(\mathbf{p}) e^{ipx}) \Big|_{p^0=E_p}. \quad (1.16)$$

The overall normalization with $(2\pi)^{-3/2}$ and the factor $2E_p$ in the integral measure are just a matter of convention at this point, because we could equally absorb them into the Fourier coefficients $a(\mathbf{p})$ and $a^*(\mathbf{p})$. Later we will find that $d^3p/(2E_p)$ defines a Lorentz-invariant integral measure, so we keep it for convenience. Furthermore, setting $p^0 = +E_p$ does not restrict us to positive-energy solutions because we would get the same form with $p^0 = -E_p$ except for the interchange $a(\mathbf{p}) \leftrightarrow a^*(-\mathbf{p})$, which we can always redefine (to see this, replace $\mathbf{p} \rightarrow -\mathbf{p}$ as integration variable). The interpretation of the positive- and negative-frequency modes $e^{\mp ipx}$ will become clear only after quantizing the theory.

Complex scalar field. We can generalize the formalism to complex scalar fields:

$$\Phi(x) = \frac{1}{\sqrt{2}} (\Phi_1(x) + i\Phi_2(x)), \quad \Phi_i^*(x) = \Phi_i(x), \quad (1.17)$$

whose Lagrangian can be written as the superposition of the Lagrangians for its real and imaginary parts:

$$\mathcal{L} = \sum_{i=1}^2 \left[\frac{1}{2} \partial_\mu \Phi_i \partial^\mu \Phi_i - \frac{m^2}{2} \Phi_i^2 \right] = \partial_\mu \Phi^* \partial^\mu \Phi - m^2 |\Phi|^2. \quad (1.18)$$

If we view the fields $\Phi(x)$ and $\Phi^*(x)$ as the independent degrees of freedom, the conjugate momenta become

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}(x)} = \dot{\Phi}^*(x), \quad \Pi^*(x) = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}^*(x)} = \dot{\Phi}(x) \quad (1.19)$$

and the Hamiltonian is

$$H = \int d^3x \left(\Pi^* \dot{\Phi} + \Pi \dot{\Phi} - \mathcal{L} \right) = \int d^3x \left(|\Pi|^2 + |\nabla \Phi|^2 + m^2 |\Phi|^2 \right). \quad (1.20)$$

Both fields satisfy Klein-Gordon equations: $(\square + m^2) \Phi = (\square + m^2) \Phi^* = 0$, and the Fourier expansion for their solutions has now the form

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} (a(\mathbf{p}) e^{-ipx} + b(\mathbf{p})^* e^{ipx}) \Big|_{p^0=E_p}, \quad (1.21)$$

with two independent coefficients $a(\mathbf{p})$ and $b(\mathbf{p})$.

(Ex)

We can define a Lorentz-invariant scalar product for solutions of the Klein-Gordon equation:

$$\langle \Phi, \Psi \rangle := i \int d\sigma_\mu \Phi^*(x) \overleftrightarrow{\partial}^\mu \psi(x) = i \int d^3x \Phi^*(x) \overleftrightarrow{\partial}_0 \psi(x), \quad (1.22)$$

where $f \overleftrightarrow{\partial}_\mu g = f(\partial_\mu g) - (\partial_\mu f)g$ and σ is a spacelike hypersurface (which we chose to be a fixed timeslice in the second step). The scalar product is Lorentz-invariant and therefore it has the same value on each spacelike hypersurface:

$$\left[\int_{\sigma_2} - \int_{\sigma_1} \right] d\sigma_\mu \Phi^* \overleftrightarrow{\partial}^\mu \Psi = \int d^4x \partial_\mu (\Phi^* \overleftrightarrow{\partial}^\mu \Psi) = \int d^4x (\Phi^* \square \Psi - \square \Phi^* \Psi) = 0. \quad (1.23)$$

In the first step we used Gauss' law under the assumption that the fields vanish sufficiently fast at $|\mathbf{x}| \rightarrow \infty$, and to obtain the zero we inserted the Klein-Gordon equations for the fields Φ and Ψ . Hence, although the fields are time-dependent, the second form in Eq. (1.22) is independent of time. Eq. (1.22) is linear in the second argument and antilinear in the first, it satisfies $\langle \Phi, \Psi \rangle^* = \langle \Psi, \Phi \rangle$, but it is not positive definite: to see this, consider the plane waves

$$f_p(x) = \frac{1}{(2\pi)^{3/2}} e^{-ipx} \Big|_{p^0=E_p} \quad \Rightarrow \quad \begin{aligned} \langle f_p, f_{p'} \rangle &= 2E_p \delta^3(\mathbf{p} - \mathbf{p}'), \\ \langle f_p^*, f_{p'}^* \rangle &= -2E_p \delta^3(\mathbf{p} - \mathbf{p}'), \\ \langle f_p, f_{p'}^* \rangle &= 0. \end{aligned} \quad (1.24)$$

With their help we can write the free Klein-Gordon solutions (1.21) as ($a_p = a(\mathbf{p})$)

$$\Phi(x) = \int \frac{d^3p}{2E_p} (a_p f_p(x) + b_p^* f_p^*(x)), \quad (1.25)$$

and therefore

$$\langle \Phi, \Phi \rangle = \int \frac{d^3p}{2E_p} \int \frac{d^3p'}{2E_{p'}} (a_p f_p + b_p^* f_p^*, a_{p'} f_{p'} + b_{p'}^* f_{p'}^*) = \int \frac{d^3p}{2E_p} (|a_p|^2 - |b_p|^2). \quad (1.26)$$

The norm is not positive definite because of the negative-energy contributions $|b_p|^2$, hence it does not permit a probability interpretation. Later we will see that $\langle \Phi | \Phi \rangle$ coincides with the $U(1)$ charge for a complex scalar field. For a real scalar field it is zero because $b_p = a_p$. From Eqs. (1.24–1.25) we can extract the Fourier coefficients via

$$a_p = \langle f_p, \Phi \rangle, \quad b_p^* = -\langle f_p^*, \Phi \rangle. \quad (1.27)$$

Noether theorem. Symmetries play a fundamental role in field theories. For example, Poincaré invariance was the guiding principle for the construction of the Lagrangian (1.10), and eventually we will see that also the properties of ‘mass’ and ‘spin’ of a particle have their origin in the Poincaré group (they are related to the Casimir operators of the group). There are also other types of symmetries such as *internal* symmetries, and generally the invariance of the action under a symmetry leads to conserved currents and charges. Symmetries also have dynamical implications: in fact, the very nature of the Standard Model as a collection of gauge theories, where charged particles interact via gauge bosons, is a consequence of *gauge invariance*.

Consider a field theory with fields $\Phi_i(x)$ and action S . We perform a transformation of the coordinates and fields, which are parametrized by infinitesimal parameters ε_a :

$$\begin{aligned} x'^\mu &= x^\mu + \delta x^\mu, & \delta x^\mu &= \sum_a \varepsilon_a X_a^\mu(x), \\ \Phi'_i(x') &= \Phi_i(x) + \delta \Phi_i, & \delta \Phi_i &= \sum_a \varepsilon_a F_{ia}(\Phi, \partial\Phi). \end{aligned} \quad (1.28)$$

The **Noether theorem** states that for each transformation that leaves the action invariant (then we call it a *symmetry transformation*) there is a conserved Noether current $j_a^\mu(x)$ with

$$\partial_\mu j_a^\mu(x) = 0 \quad (1.29)$$

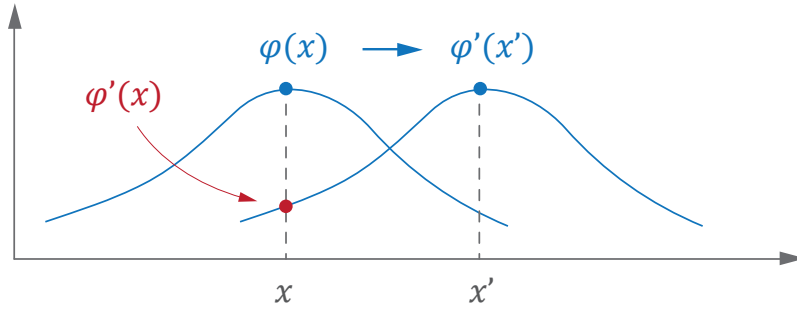


FIGURE 1.1: Visualization of Eqs. (1.33–1.34).

along the classical trajectories, i.e., for solutions of the classical equations of motion. We note that one can still write down a Noether current $j_a^\mu(x)$ irrespective of whether the transformation (1.28) is a symmetry or not (then it won't be conserved), and in general we do not require the fields $\Phi_i(x)$ to satisfy the classical equations of motion.

Here are some examples for symmetry transformations:

- **Internal symmetries** correspond to transformations of the fields only, but not spacetime itself. They are usually realized in the form of Lie groups whose elements are obtained by exponentiating the group generators G_a :

$$\Phi'_i(x) = D_{ij} \Phi_j(x), \quad D = e^{i \sum_a \varepsilon_a G_a} \quad \Leftrightarrow \quad \begin{aligned} \delta x^\mu &= 0, \\ \delta \Phi_i &= i \sum_a \varepsilon_a (G_a)_{ij} \Phi_j. \end{aligned} \quad (1.30)$$

- Spacetime **translations** depend on four parameters a^μ and they are part of the Poincaré group:

$$\begin{aligned} x' &= x + a \\ \Phi'_i(x + a) &= \Phi_i(x) \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \delta x^\mu &= a^\mu \\ \delta \Phi_i &= 0. \end{aligned} \quad (1.31)$$

- **Lorentz transformations** consist of rotations and boosts and contain the remaining six parameters of the Poincaré group. An infinitesimal Lorentz transformation $\Lambda = 1 + \varepsilon$ is parametrized by the antisymmetric matrix $\varepsilon_{\mu\nu}$:

$$\begin{aligned} x' &= \Lambda x \\ \Phi'_i(\Lambda x) &= D_{ij}(\Lambda) \Phi_j(x) \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \delta x^\mu &= \varepsilon^{\mu\nu} x_\nu \\ \delta \Phi_i &= \dots \end{aligned} \quad (1.32)$$

The matrices $D(\Lambda)$ are the finite-dimensional irreducible representations of the Lorentz group which depend on the nature of the fields (scalar, Dirac, vector field etc.); we will discuss them later in the context of Dirac theory. For scalar fields, $D(\Lambda) = 1$ and therefore they satisfy $\Phi'_i(\Lambda x) = \Phi_i(x)$ and $\delta \Phi_i = 0$ (which is why the fields are *scalars* under Lorentz transformations).

To proceed, we need to define two types of variations. The ‘total’ variation is what we already introduced above:

$$\delta\Phi_i = \Phi'_i(x') - \Phi_i(x). \quad (1.33)$$

It vanishes for the example of a scalar field under Poincaré transformations. The second type of variation is the change of the *functional form* of the field at the position x :

$$\begin{aligned} \delta_0\Phi_i &= \Phi'_i(x) - \Phi_i(x) \\ &= \Phi'_i(x' - \delta x) - \Phi_i(x) = \Phi'_i(x') - \partial_\mu\Phi_i \delta x^\mu - \Phi_i(x) \\ &= \delta\Phi_i - \partial_\mu\Phi_i \delta x^\mu. \end{aligned} \quad (1.34)$$

Both types of variations are visualized in Fig. 1.1: a scalar field is invariant under translations and therefore $\Phi'(x') = \Phi(x)$; however, the functional form $\Phi'(x)$ at the position x has changed in the process. It follows that

$$\delta\Phi_i = \delta_0\Phi_i + \partial_\mu\Phi_i \delta x^\mu, \quad (1.35)$$

where the second term vanishes for internal symmetries ($\delta x^\mu = 0$).

Consider now a variation of the action of the form

$$\begin{aligned} \delta S &= \int d^4x' \mathcal{L}(\Phi'(x'), \partial'_\mu\Phi'(x')) - \int d^4x \mathcal{L}(\Phi(x), \partial_\mu\Phi(x)) \\ &= \int d^4x \delta\mathcal{L} + \int (\delta d^4x) \mathcal{L}, \end{aligned} \quad (1.36)$$

with $\partial'_\mu = \partial/\partial x'^\mu$, which does not vanish at the boundary and also permits a variation of the volume itself. The variation of the integral measure follows from expanding the Jacobian of the transformation:

$$d^4x' = |\det J| d^4x = (1 + \partial_\mu\delta x^\mu + \dots) d^4x \quad \Rightarrow \quad \delta d^4x = (\partial_\mu\delta x^\mu) d^4x. \quad (1.37)$$

Inserting this together with Eq. (1.35) into the expression for δS , we get

$$\begin{aligned} \delta S &= \int d^4x [\delta_0\mathcal{L} + \partial_\mu\mathcal{L} \delta x^\mu + \mathcal{L} \partial_\mu\delta x^\mu] \\ &= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\Phi} \delta_0\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)} \delta_0\partial_\mu\Phi + \partial_\mu(\mathcal{L} \delta x^\mu) \right] \\ &= \int_V d^4x \left\{ \underbrace{\left[\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)} \right]}_{\text{eqs. of motion}} \delta_0\Phi + \partial_\mu \left[\underbrace{\mathcal{L} \delta x^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)} \delta_0\Phi}_{-\delta j^\mu = -\sum_a \varepsilon_a j_a^\mu} \right] \right\}. \end{aligned} \quad (1.38)$$

In the second bracket we defined a current δj^μ ; it inherits the dependence on the infinitesimal transformation parameters ε_a in Eq. (1.28), so there is one current j_a^μ for each parameter ε_a . Now, if these transformations define a symmetry of the action then $\delta S = 0$, and because the spacetime volume is arbitrary also the integrand must be

zero. The first bracket vanishes upon inserting the solutions of the classical equations of motion, and so we arrive at a conserved Noether current for each ε_a :

$$\partial_\mu j_a^\mu(x) = 0. \quad (1.39)$$

We can rewrite the Noether current in a more useful form. With Eq. (1.35) we eliminate $\delta_0\Phi$ in favor of the total variation $\delta\Phi$:

$$-\delta j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)} \delta\Phi - \underbrace{\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)} \partial^\nu\Phi - g^{\mu\nu}\mathcal{L} \right]}_{=: T^{\mu\nu}} \delta x_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)} \delta\Phi - T^{\mu\nu} \delta x_\nu. \quad (1.40)$$

$T^{\mu\nu}$ is the **energy-momentum tensor** whose meaning will become clear in a moment. While we derived the Noether theorem and the current for a single-component field, the derivation goes through for arbitrary types of fields in arbitrary representations of the Lorentz group — one simply has to sum over all fields in the Lagrangian. Let's exemplify the case of a scalar field $\Phi(x)$ under ...

- translations ($\delta\Phi = 0$, $\delta x_\nu = a_\nu$): the first term in δj^μ vanishes, and after removing the translation parameters a^ν we find that the conserved current according to translation invariance is the energy-momentum tensor itself. The continuity equation

$$\partial_\mu T^{\mu\nu} = 0 \quad (1.41)$$

holds for solutions of the Klein-Gordon equation and can be easily verified. The energy-momentum tensor has the form $T^{\mu\nu} = \partial^\mu\Phi \partial^\nu\Phi - g^{\mu\nu}\mathcal{L}$, which corresponds to one current for each component of a^ν .

- Lorentz transformations ($\delta\Phi = 0$, $\delta x_\alpha = \varepsilon_{\alpha\beta} x^\beta$): here we can exploit the anti-symmetry of $\varepsilon_{\alpha\beta}$ and write

$$-\delta j^\mu = -T^{\mu\alpha} \varepsilon_{\alpha\beta} x^\beta = -\frac{\varepsilon_{\alpha\beta}}{2} (T^{\mu\alpha} x^\beta - T^{\mu\beta} x^\alpha) =: -\frac{\varepsilon_{\alpha\beta}}{2} m^{\mu,\alpha\beta}. \quad (1.42)$$

Therefore, the conserved current is the **angular momentum density**

$$m^{\mu,\alpha\beta} = T^{\mu\alpha} x^\beta - T^{\mu\beta} x^\alpha, \quad \partial_\mu m^{\mu,\alpha\beta} = 0. \quad (1.43)$$

It carries the orbital angular momentum of the field; for fields with higher spin there will be additional spin contributions coming from the $\delta\Phi$ term in Eq. (1.40). We can make this more explicit by inserting the energy-momentum tensor:

$$m^{\mu,\alpha\beta} = -i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)} \underbrace{\left[-i(x^\alpha \partial^\beta - x^\beta \partial^\alpha) \right]}_{=: L^{\alpha\beta}} \Phi + (x^\alpha g^{\mu\beta} - x^\beta g^{\mu\alpha}) \mathcal{L}. \quad (1.44)$$

$L^{\alpha\beta}$ is a 'covariantized' version of the orbital angular momentum, because in analogy to Eq. (2.55) we can define a three-vector

$$L^i := -\frac{1}{2} \varepsilon_{ijk} L^{jk} = i \varepsilon_{ijk} x^j \partial^k \quad \Rightarrow \quad \mathbf{L} = \mathbf{x} \times (-i\nabla). \quad (1.45)$$

For a scalar field the current has the explicit form

$$m^{\mu,\alpha\beta} = -i \partial^\mu\Phi L^{\alpha\beta} \Phi + (x^\alpha g^{\mu\beta} - x^\beta g^{\mu\alpha}) \mathcal{L}. \quad (1.46)$$

- Internal symmetries ($\delta x_\nu = 0$): in that case only the first term in Eq. (1.40) contributes. An example is the $U(1)$ current of a complex scalar field that we will discuss in Eq. (2.44). The Lagrangian is invariant under the transformation $\Phi' = e^{i\varepsilon}\Phi$, $\Phi'^* = e^{-i\varepsilon}\Phi^*$, with ε a real constant, and the corresponding current is

$$j^\mu = i(\Phi^* \partial^\mu \Phi - \partial^\mu \Phi^* \Phi) = i \Phi^* \overleftrightarrow{\partial}^\mu \Phi, \quad \partial_\mu j^\mu = 0. \quad (1.47)$$

Noether charges. There is another important consequence of current conservation. After inserting the equations of motion into Eq. (1.38), we can exploit Gauss' law to convert the remaining volume integral into a surface integral:

$$0 = \int_V d^4x \partial_\mu j_a^\mu = \int_{\partial V} d\sigma_\mu j_a^\mu. \quad (1.48)$$

Specifically, if we squeeze the spacetime volume between two hypersurfaces at fixed times, $d\sigma_\mu = (d^3x, \mathbf{0})$, and assume that the fields vanish sufficiently fast for $|\mathbf{x}| \rightarrow \infty$, we conclude that there is a conserved charge that has the same value for all times:

$$\int d^3x j_a^0 \Big|_{t_2} - \int d^3x j_a^0 \Big|_{t_1} = 0 \quad \Rightarrow \quad Q_a := \int d^3x j_a^0(x) = \text{const} \quad \forall t. \quad (1.49)$$

The relative sign comes from the fact that the normal vectors for the planes at fixed times always point outwards of the volume. In fact, this relation holds for arbitrary spacelike hypersurfaces, so the charge Q_a has the same value for all spacelike surfaces.

As an example, let's consider again a scalar field under translations. The conserved current is $T^{\mu\nu}$, and therefore the conserved charges are the spatial integrals $\int d^3x T^{0\nu}$. They coincide with the Hamiltonian H and the total momentum \mathbf{P} of the system, which are conserved:

$$\begin{aligned} H &= \int d^3x T^{00} = \int d^3x (\Pi \dot{\Phi} - \mathcal{L}) = \int d^3x \mathcal{H}, \\ P^i &= \int d^3x T^{0i} = \int d^3x \Pi \partial^i \Phi = - \int d^3x \Pi \nabla_i \Phi. \end{aligned} \quad (1.50)$$

Taken together with the Lorentz transformations, the conserved charges are the quantities

$$P^\alpha = \int d^3x T^{0\alpha}, \quad M^{\alpha\beta} = \int d^3x m^{0,\alpha\beta}. \quad (1.51)$$

For example, in the case of rotations the conserved charge is

$$J^i = -\frac{1}{2} \varepsilon_{ijk} M^{jk} = -\frac{1}{2} \varepsilon_{ijk} \int d^3x m^{0,jk} \stackrel{(1.45)}{=} \frac{i}{2} \varepsilon_{ijk} \int d^3x \Pi L^{jk} \Phi, \quad (1.52)$$

which is the total angular momentum of the field:

$$\mathbf{J} = -i \int d^3x \Pi \mathbf{L} \Phi. \quad (1.53)$$

Another example is the conserved charge for the $U(1)$ current in Eq. (1.47):

$$Q = i \int d^3x \Phi^* \overleftrightarrow{\partial}_0 \Phi. \quad (1.54)$$

This is just our earlier construction of the ‘norm’ for Klein-Gordon solutions; it is indeed Lorentz-invariant because it has the same value on each spacelike hypersurface.

The Noether charges will play a prominent role in the quantum field theory. After quantizing the fields by imposing commutator relations, the charges inherit the operator structure of the fields and form a representation of the Lie algebra of the symmetry group on the Fock space. That is, if the group elements of the symmetry transformation in Eq. (1.28) can be written as

$$D = e^{i \sum_a \varepsilon_a G_a} \quad \text{with} \quad [G_a, G_b] = i f_{abc} G_c, \quad (1.55)$$

with some generic structure constants f_{abc} , then the charges will satisfy the same Lie-algebra relation as the generators:

$$[Q_a, Q_b] = i f_{abc} Q_c, \quad (1.56)$$

and thereby provide a representation of the symmetry group on the state space. This is also true for the Poincaré group (which is also a Lie group): after quantization, the operators $M^{\mu\nu}$ and P^μ in Eq. (1.51) satisfy the commutator relations of the Poincaré algebra and thereby form a unitary representation of the Poincaré group on the state space.

2 Quantization of the scalar field

Commutator relations. The strategy to quantize a classical field theory is to interpret the fields $\Phi(x)$ and $\Pi(x) = \dot{\Phi}(x)$ as operators which satisfy canonical commutation relations. This is completely analogous to the transition from classical to quantum mechanics for discrete systems, where q_i and p_i are promoted to self-adjoint operators that satisfy

$$[q_i, p_j] = i\delta_{ij}, \quad [q_i, q_j] = [p_i, p_j] = 0. \quad (2.1)$$

These relations hold in the Schrödinger picture where the time dependence is carried by the states alone; in the Heisenberg picture the operators are time-dependent and the commutation relations are imposed at equal times. In the following we will always work in the Heisenberg picture, so we demand that for equal times

$$\begin{aligned} [\Phi(x), \Pi(y)]_{x^0=y^0} &= i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \\ [\Phi(x), \Phi(y)]_{x^0=y^0} &= [\Pi(x), \Pi(y)]_{x^0=y^0} = 0. \end{aligned} \quad (2.2)$$

Despite appearances, this does not destroy Lorentz covariance because x and y are separated by a spacelike distance $(x - y)^2 < 0$ which is preserved under a Lorentz transformation. By virtue of the Dirac delta function, $\Phi(x)$ and $\Pi(x)$ are now operator-valued distributions; to arrive at well-defined expressions one should in principle ‘smear’ them with smooth test functions.

The wave functions in quantum mechanics are also fields $\Phi(x)$ that satisfy (relativistic or non-relativistic) wave equations, but there they are interpreted as single-particle wave functions in some Hilbert space by imposing an appropriate scalar product. (Unfortunately, already for relativistic Klein-Gordon particles the scalar product is not positive definite, so we lost the probability interpretation). In quantum field theory we impose instead an operator structure on $\Phi(x)$, which is why the procedure is often called ‘second quantization’. Since we really only quantize the field $\Phi(x)$ once, the correct term should be ‘field quantization’.

Fourier expansion. We write the Fourier expansion for solutions of the free Klein-Gordon equation as

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} \left(a(\mathbf{p}) e^{-ipx} + a^\dagger(\mathbf{p}) e^{ipx} \right) \Big|_{p^0=E_p}, \quad (2.3)$$

so the Fourier coefficients (from now on we abbreviate $a(\mathbf{p}) \equiv a_p$) will inherit the operator structure. In the following we will often encounter the Lorentz-invariant integral measure $\int d^3p/(2E_p)$ that is obtained by restricting the four-momentum integration to the positive-energy mass shell (which is a Lorentz-invariant condition):

$$\int d^4p \Theta(p^0) \delta(p^2 - m^2) = \int d^4p \Theta(p^0) \frac{\delta(p^0 - E_p) + \delta(p^0 + E_p)}{2E_p} = \int \frac{d^3p}{2E_p}. \quad (2.4)$$

Consequently, also the combination $2E_p \delta^3(\mathbf{p} - \mathbf{p}')$ is Lorentz-invariant. Upon inserting the Fourier expansion into Eq. (2.2) we obtain the commutation relations for $a_p, a_{p'}^\dagger$:

$$[a_p, a_{p'}^\dagger] = 2E_p \delta^{(3)}(\mathbf{p} - \mathbf{p}'), \quad [a_p, a_{p'}] = [a_p^\dagger, a_{p'}^\dagger] = 0. \quad (2.5)$$

This can be shown in several ways. For example, let's write

$$\begin{aligned}\Phi(x) &= \frac{1}{(2\pi)^{3/2}} \int d^3p \left(\frac{a_p e^{-iE_p t} + a_{-p}^\dagger e^{iE_p t}}{2E_p} \right) e^{i\mathbf{p}\cdot\mathbf{x}}, \\ \Pi(x) = \dot{\phi}(x) &= \frac{1}{(2\pi)^{3/2}} \int d^3p \left(\frac{a_p e^{-iE_p t} - a_{-p}^\dagger e^{iE_p t}}{2i} \right) e^{i\mathbf{p}\cdot\mathbf{x}},\end{aligned}\tag{2.6}$$

and abbreviate the two brackets (the three-dimensional Fourier transforms) by

$$\tilde{\Phi}_p(t) = \frac{1}{2E_p} (a_p(t) + a_{-p}^\dagger(t)), \quad \tilde{\Pi}_p(t) = \frac{1}{2i} (a_p(t) - a_{-p}^\dagger(t)).\tag{2.7}$$

It follows that

$$\begin{aligned}\tilde{\Phi}_p^\dagger(t) &= \tilde{\Phi}_{-p}(t), & a_p(t) &= E_p \tilde{\Phi}_p(t) + i \tilde{\Pi}_p(t), \\ \tilde{\Pi}_p^\dagger(t) &= \tilde{\Pi}_{-p}(t), & a_p^\dagger(t) &= E_p \tilde{\Phi}_p^\dagger(t) - i \tilde{\Pi}_p^\dagger(t).\end{aligned}\tag{2.8}$$

Now insert this into the commutator:

$$[\Phi(x), \pi(y)]_{x^0=y^0} = \int \frac{d^3p d^3p'}{(2\pi)^3} e^{i(\mathbf{p}\cdot\mathbf{x} - \mathbf{p}'\cdot\mathbf{y})} [\tilde{\Phi}_p(t), \tilde{\Pi}_{p'}^\dagger(t)] \stackrel{!}{=} i\delta^{(3)}(\mathbf{x} - \mathbf{y}).\tag{2.9}$$

Here we have changed the integration variable from $\mathbf{p} \rightarrow -\mathbf{p}$ and used $\tilde{\Pi}_{-p}(t) = \tilde{\Pi}_p^\dagger(t)$. Hence, the commutator for the Fourier transformed quantities must be also a δ -function (the time dependence cancels),

$$[\tilde{\Phi}_p(t), \tilde{\Pi}_{p'}^\dagger(t)] = i\delta^{(3)}(\mathbf{p} - \mathbf{p}'),\tag{2.10}$$

and we can extract the commutator relation for a_p and $a_{p'}^\dagger$:

$$[a_p, a_{p'}^\dagger] = [a_p(t), a_{p'}^\dagger(t)] = -iE_p [\tilde{\Phi}_p(t), \tilde{\Pi}_{p'}^\dagger(t)] + iE_{p'} [\tilde{\Phi}_{p'}^\dagger(t), \tilde{\Pi}_p^\dagger(t)]^\dagger = 2E_p \delta^{(3)}(\mathbf{p} - \mathbf{p}').\tag{2.11}$$

Another way to arrive at this result is to use Eq. (1.27) for the Fourier coefficients and calculate the commutator directly:

$$a_p = \langle f_p, \Phi \rangle = i \int d^3x f_p^*(x) \overleftrightarrow{\partial}_0 \Phi(x) \Big|_{x^0=t}, \quad a_p^\dagger = -\langle f_p^\dagger, \Phi \rangle = -i \int d^3x f_p(x) \overleftrightarrow{\partial}_0 \Phi(x) \Big|_{x^0=t}.\tag{2.12}$$

For equal times $x^0 = y^0 = t$ we can insert the commutator relations (2.2), so that

$$\begin{aligned}[a_p, a_{p'}^\dagger] &= \int d^3x \int d^3y f_p^*(x) f_{p'}(y) \overleftrightarrow{\partial}_{x^0} \overleftrightarrow{\partial}_{y^0} [\Phi(x), \Phi(y)] \doteq \\ &= \int d^3x \int d^3y \left[f_p^*(x) \dot{f}_{p'}(y) - \dot{f}_p^*(x) f_{p'}(y) \right] i\delta^3(\mathbf{x} - \mathbf{y}) \\ &= i \int d^3x f_p^*(x) \overleftrightarrow{\partial}_0 f_{p'}(x) = \langle f_p, f_{p'} \rangle = 2E_p \delta^3(\mathbf{p} - \mathbf{q}).\end{aligned}\tag{2.13}$$

Hamilton and momentum operator. To proceed, we derive the Fourier decomposition for the Hamiltonian (1.15) of the free scalar field theory. The form of the Hamiltonian already resembles that of a collection of harmonic oscillators at each point \mathbf{x} , but the term $(\nabla\Phi)^2$ couples the degrees of freedom at \mathbf{x} and $\mathbf{x} + \delta\mathbf{x}$. We can diagonalize it in momentum space by inserting the relations (2.6–2.7); in that way it becomes the sum of decoupled harmonic oscillators with frequencies E_p :

$$H = \int d^3x \frac{1}{2} [\Pi^2 + (\nabla\Phi)^2 + m^2\Phi^2] = \int d^3p \frac{1}{2} \left[\tilde{\Pi}_p^\dagger(t) \tilde{\Pi}_p(t) + E_p^2 \tilde{\Phi}_p^\dagger(t) \tilde{\Phi}_p(t) \right].\tag{2.14}$$

To arrive at this result, use

$$\begin{aligned}\int d^3x (\nabla\Phi)^2 &= \int d^3x \int \frac{d^3p d^3p'}{(2\pi)^3} \tilde{\Phi}_p(t) \tilde{\Phi}_{p'}(t) (-\mathbf{p} \cdot \mathbf{p}') e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} \\ &= \int d^3p \tilde{\Phi}_p(t) \tilde{\Phi}_{-p}(t) \mathbf{p}^2 = \int d^3p \tilde{\Phi}_p^\dagger(t) \tilde{\Phi}_p(t) \mathbf{p}^2\end{aligned}\tag{2.15}$$

and similarly

$$\int d^3x \Phi^2 = \int d^3p \tilde{\Phi}_p^\dagger(t) \tilde{\Phi}_p(t), \quad \int d^3x \Pi^2 = \int d^3p \tilde{\Pi}_p^\dagger(t) \tilde{\Pi}_p(t). \quad (2.16)$$

Inserting the decomposition (2.7) finally yields the result

$$H = \int \frac{d^3p}{2E_p} E_p \frac{a_p^\dagger a_p + a_p a_p^\dagger}{2}. \quad (2.17)$$

Unfortunately this expression is divergent because it contains the sum of the zero-point energy of all oscillators:

$$\frac{a_p^\dagger a_p + a_p a_p^\dagger}{2} = a_p^\dagger a_p + \frac{1}{2} [a_p, a_p^\dagger] = a_p^\dagger a_p + E_p \delta^3(\mathbf{0}). \quad (2.18)$$

The Dirac delta is proportional to the volume; had we studied the system in a finite box, we would write $(2\pi)^3 \delta^3(\mathbf{0}) \rightarrow V$. (This is an **infrared divergence**.) However, for large \mathbf{p} we have $E_p \sim \sqrt{\mathbf{p}^2 + m^2} \simeq |\mathbf{p}|$ and the integral still diverges. If we regulate the divergence by integrating only up to a cutoff $|\mathbf{p}| \leq \Lambda$, the energy density of the vacuum becomes

$$\rho_{\text{vac}} = \frac{E_{\text{vac}}}{V} = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} E_p \sim \int dp p^3 \sim \Lambda^4. \quad (2.19)$$

This is a first example of an **ultraviolet divergence** which we will frequently encounter later. Since (in a theory without gravity) we can only measure energy differences, we can simply discard it so that the vacuum energy is zero. This is formally called *normal ordering* or *Wick ordering*: we obtain the normal-ordered form $:\mathcal{O}:$ of some operator \mathcal{O} by moving all creation operators to the left of all destruction operators. Later when we discuss renormalization we will see how UV divergences can be systematically removed from the theory; for the time being we interpret all operators as being normal-ordered. Hence, the Hamilton operator becomes

$$H = \int d^3x \frac{1}{2} :[\Pi^2 + (\nabla\Phi)^2 + m^2\Phi^2]: = \int \frac{d^3p}{2E_p} E_p a_p^\dagger a_p. \quad (2.20)$$

We can repeat the procedure to obtain the spatial momentum operator \mathbf{P} . We identify it with the classical charge $P^i = \int d^3x T^{0i}$ in Eq. (1.50) that follows from the invariance under spatial translations. The analogous calculation gives

$$\mathbf{P} = - \int d^3x : \Pi \nabla \Phi : = i \int d^3p \mathbf{p} : \tilde{\Pi}_p(t) \tilde{\Phi}_p^\dagger(t) : = \int \frac{d^3p}{2E_p} \mathbf{p} a_p^\dagger a_p, \quad (2.21)$$

so that we can combine Eqs. (2.20) and (2.21) into the covariant four-momentum operator

$$P^\mu = \int d^3x : T^{0\mu} : = \int \frac{d^3p}{2E_p} p^\mu a_p^\dagger a_p \Big|_{p^0=E_p}. \quad (2.22)$$

Fock space. What is the Hilbert space on which the four-momentum operator acts? Since P^μ is self-adjoint it has eigenstates with real eigenvalues. Let $|k\rangle$ be such an eigenstate with $P^\mu |k\rangle = k^\mu |k\rangle$, so that \mathbf{k} is the momentum of the state and $k^0 = E_k$ its energy. First we observe that the energy (and therefore H itself) is non-negative:

$$\langle \lambda | H | \lambda \rangle = \int \frac{d^3 p}{2E_p} E_p \langle \lambda | a_p^\dagger a_p | \lambda \rangle \geq 0, \quad (2.23)$$

because the integrand is $||a_p | \lambda \rangle||^2$. On the other hand, we can calculate the commutators

$$[P^\mu, a_q^\dagger] = q^\mu a_q^\dagger, \quad [P^\mu, a_q] = -q^\mu a_q \quad (2.24)$$

and use them to show that if $|k\rangle$ is an eigenstate of P^μ , then also $a_q^\dagger |k\rangle$ and $a_q |k\rangle$ are eigenstates of P^μ with their eigenvalues shifted by the momentum $\pm q^\mu$:

$$\begin{aligned} P^\mu (a_q^\dagger |k\rangle) &= a_q^\dagger (P^\mu + q^\mu) |k\rangle = (k + q)^\mu a_q^\dagger |k\rangle, \\ P^\mu (a_q |k\rangle) &= a_q (P^\mu - q^\mu) |k\rangle = (k - q)^\mu a_q |k\rangle, \end{aligned} \quad (2.25)$$

which at the same time shifts the energy of the state. Hence we can interpret a_q^\dagger, a_q as ladder operators. Since the total energy cannot be smaller than zero, there must be a state with $a_q |0\rangle = 0 \forall \mathbf{q}$, because otherwise the successive action of a_q would lead to negative eigenvalues of H .

We call $|0\rangle$ the vacuum of the theory. It has four-momentum zero: $P^\mu |0\rangle = 0$, and we normalize it to $\langle 0|0\rangle = 1$. The state $a_k^\dagger |0\rangle$ then has four-momentum $k^\mu = (E_k, \mathbf{k})$:

$$P^\mu a_k^\dagger |0\rangle = k^\mu a_k^\dagger |0\rangle. \quad (2.26)$$

Since $E_k = \sqrt{\mathbf{k}^2 + m^2}$ is the relativistic dispersion relation for a single particle with mass m , we interpret $|k\rangle = a_k^\dagger |0\rangle$ as a one-particle state with energy E_k and momentum \mathbf{k} . Its normalization is

$$\langle k | k' \rangle = \langle 0 | a_k a_{k'}^\dagger | 0 \rangle = \langle 0 | a_{k'}^\dagger a_k + 2E_k \delta^3(\mathbf{k} - \mathbf{k}') | 0 \rangle = 2E_k \delta^3(\mathbf{k} - \mathbf{k}') \quad (2.27)$$

which, in turn, leads to the Lorentz-invariant completeness relation on the one-particle Hilbert space:

$$\mathbb{1}_{\text{1-particle}} = \int \frac{d^3 k}{2E_k} |k\rangle \langle k|. \quad (2.28)$$

Similarly, for a two-particle state we obtain

$$P^\mu a_q^\dagger a_k^\dagger |0\rangle = (q^\mu + k^\mu) a_q^\dagger a_k^\dagger |0\rangle \quad (2.29)$$

and so on. A generic N -particle state has the form $|k_1 \dots k_N\rangle = a_{k_1}^\dagger \dots a_{k_N}^\dagger |0\rangle$, and the eigenvalue of the momentum operator is the total momentum of the system:

$$P^\mu |k_1 \dots k_N\rangle = (k_1^\mu + \dots + k_N^\mu) |k_1 \dots k_N\rangle. \quad (2.30)$$

The resulting *Fock space* is the direct sum of all N -particle Hilbert spaces ($N \in \mathbb{N}_0$). From the fact that the creation operators commute between themselves we also see that

these multiparticle states are symmetric under the exchange of any two particles, so they obey Bose–Einstein statistics. This is an example of the spin–statistics theorem, which states that particles with integer spin are bosons and particles with half-integer spin are fermions.

Generally, a multiparticle state that contains K different momenta k_i , $i = 1 \dots K$, with $n(k_i)$ particles carrying momentum k_i and $\sum_{i=1}^K n(k_i)$ particles in total, can be written as

$$|n(k_1) n(k_2) \dots n(k_K)\rangle = \prod_{i=1}^K \frac{(a_{k_i}^\dagger)^{n(k_i)}}{\sqrt{n(k_i)!}} |0\rangle, \quad (2.31)$$

where the denominator takes care of multiplicities in the same momentum. We can count the total number of particles in such a state with the number operator

$$N = \int \frac{d^3p}{2E_p} a_p^\dagger a_p. \quad (2.32)$$

The eigenvalues of the operators N and P^μ are the total number of particles and the total four-momentum, respectively:

$$N \rightarrow \sum_{i=1}^K n(k_i), \quad P^\mu \rightarrow \sum_{i=1}^K n(k_i) k_i^\mu. \quad (2.33)$$

This can be easily proven for $K = 1$, i.e., for a state $|n(k)\rangle$ that consists of $n(k)$ identical particles with momentum k : simply commute a_p in Eqs. (2.22) and (2.32) to the right until it annihilates on the vacuum. The eigenvalue of N is $n(k)$ and the total momentum is $n(k) k^\mu$, and therefore the total energy $n(k) E_k$ is the sum of the energies of all particles.

We can now also better understand the meaning of the field $\Phi(x)$. Written in Fourier modes (2.3) and acting on the vacuum, it creates a particle at the position x :

$$\Phi(x)|0\rangle = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} e^{ipx} |p\rangle \Big|_{p^0=E_p} = |x\rangle, \quad (2.34)$$

and with the normalization (2.27) we can write the one-particle ‘wave function’ as the overlap

$$\langle x|p\rangle = \langle 0|\Phi(x)|p\rangle = \frac{1}{(2\pi)^{3/2}} e^{-ipx} \Big|_{p^0=E_p}. \quad (2.35)$$

In that way the fundamental entities in quantum field theory are not the particles but rather the field $\Phi(x)$ which penetrates spacetime. Although it is not measurable by itself, we can interpret it as the ‘property of spacetime’ to create particles of momentum \mathbf{p} and energy E_p as its excitations.

Complex scalar field and antiparticles. Let’s generalize the formalism to complex scalar fields $\Phi(x)$ and $\Phi^\dagger(x)$, because this will allow us to describe not only particles but also their antiparticles. The Lagrangian has the form

$$\mathcal{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi, \quad (2.36)$$

the conjugate momenta become

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}(x)} = \dot{\Phi}^\dagger(x), \quad \Pi^\dagger(x) = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}^\dagger(x)} = \dot{\Phi}(x) \quad (2.37)$$

and the Hamiltonian is

$$H = \int d^3x \left(\Pi^\dagger \dot{\Phi}^\dagger + \Pi \dot{\Phi} - \mathcal{L} \right) = \int d^3x \left(\Pi^\dagger \Pi + \nabla \Phi^\dagger \nabla \Phi + m^2 \Phi^\dagger \Phi \right). \quad (2.38)$$

The commutator relations are

$$\left[\Phi(x), \Pi(y) \right]_{x^0=y^0} = \left[\Phi^\dagger(x), \Pi^\dagger(y) \right]_{x^0=y^0} = i\delta^3(\mathbf{x} - \mathbf{y}) \quad (2.39)$$

whereas all other commutators vanish. The Fourier expansion has now the form

$$\begin{aligned} \Phi(x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} \left(a_p e^{-ipx} + b_p^\dagger e^{ipx} \right) \Big|_{p^0=E_p}, \\ \Phi^\dagger(x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} \left(b_p e^{-ipx} + a_p^\dagger e^{ipx} \right) \Big|_{p^0=E_p} \end{aligned} \quad (2.40)$$

with independent operators a_p and b_p whose commutation relations become

$$[a_p, a_{p'}^\dagger] = [b_p, b_{p'}^\dagger] = 2E_p \delta^3(\mathbf{p} - \mathbf{p}'), \quad (2.41)$$

with all others zero. The mode expansion of the four-momentum operator is

$$P^\mu = \int \frac{d^3p}{2E_p} p^\mu \left(a_p^\dagger a_p + b_p^\dagger b_p \right) \quad (2.42)$$

and implies that there are now two types of particles, with two types of momentum eigenstates $a_p^\dagger |0\rangle$ and $b_p^\dagger |0\rangle$ which have the same momentum \mathbf{p} , energy $E_p = \sqrt{\mathbf{p}^2 + m^2}$ and mass m . Since they are scalar particles they also have both spin zero. So what distinguishes them?

There is a new property that is particular to the Lagrangian for a complex scalar field: is invariant under the continuous global $U(1)$ symmetry

$$\Phi'(x) = e^{i\varepsilon} \Phi(x), \quad \Phi'^\dagger(x) = e^{-i\varepsilon} \Phi^\dagger(x), \quad (2.43)$$

with $\varepsilon \in \mathbb{R}$ constant. According to Noether's theorem there is now a conserved current

$$j^\mu = i : (\Phi^\dagger \partial^\mu \Phi - \partial^\mu \Phi^\dagger \Phi) : \quad (2.44)$$

whose corresponding charge is

$$Q = \int d^3x j^0 = i \int d^3x : (\Phi^\dagger \dot{\Phi} - \dot{\Phi}^\dagger \Phi) : = \int \frac{d^3p}{2E_p} \left(a_p^\dagger a_p - b_p^\dagger b_p \right). \quad (2.45)$$

Comparing this with Eq. (2.32), we see that the $U(1)$ charge describes the total number of particles created by a_p^\dagger minus that created by b_p^\dagger , which is conserved. For example, its eigenvalues for one-particle states are

$$Q a_p^\dagger |0\rangle = a_p^\dagger |0\rangle, \quad Q b_p^\dagger |0\rangle = -b_p^\dagger |0\rangle. \quad (2.46)$$

We will call them particles and antiparticles; for the real field the particle is its own antiparticle. Now we can also interpret the negative-energy solutions of the Klein-Gordon equation: via Eq. (2.40) the coefficient of the positive-energy solution e^{-ipx}

becomes the annihilation operator of a particle and that of e^{ipx} the creation operator of its antiparticle. In the context of QED we will later find that the $U(1)$ Noether charge indeed corresponds to the *electric* charge, i.e., the coupling to the electromagnetic field.

Poincaré algebra. In our discussion of the Poincaré group we saw that Poincaré transformations have the form

$$x' = T(\Lambda, a)x = \Lambda x + a \quad \Leftrightarrow \quad \delta x^\mu = \varepsilon^{\mu\nu} x_\nu + a^\mu. \quad (2.47)$$

The group axioms are satisfied: the transformation is associative, $(TT')T'' = T(T'T'')$, the unit element is $T(1, 0)$, two consecutive Poincaré transformations form another one: $T(\Lambda', a')T(\Lambda, a) = T(\Lambda'\Lambda, a' + \Lambda'a)$, and from equating this with $T(1, 0)$ we can read off the inverse element: $T^{-1}(\Lambda, a) = T(\Lambda^{-1}, -\Lambda^{-1}a)$.

Consider now the representations $U(\Lambda, a)$ of the Poincaré group on some vector space. They inherit the group structure from the $T(\Lambda, a)$, and we use the symbol U although they are not necessarily unitary. The Poincaré group $ISO(3, 1)^\uparrow$ is a Lie group and therefore its elements can be written as

$$U(\Lambda, a) = e^{\frac{i}{2}\varepsilon_{\mu\nu} M^{\mu\nu}} e^{ia_\mu P^\mu} = 1 + \frac{i}{2}\varepsilon_{\mu\nu} M^{\mu\nu} + ia_\mu P^\mu + \dots, \quad (2.48)$$

with infinitesimal generators $M^{\mu\nu}$ for Lorentz transformations and P^μ for translations. Their explicit form depends on the representation, i.e., it is determined by the vector space on which they act. Since $\varepsilon_{\mu\nu}$ is totally antisymmetric, $M^{\mu\nu}$ can also be chosen antisymmetric. It contains the six generators of the Lorentz group, whereas the momentum operator P^μ is the generator of spacetime translations. $M^{\mu\nu}$ and P^μ form a Lie algebra (the Poincaré algebra) whose commutator relations are given by

$$i[M^{\mu\nu}, M^{\rho\sigma}] = g^{\mu\sigma} M^{\nu\rho} + g^{\nu\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\nu\sigma} - g^{\nu\sigma} M^{\mu\rho}, \quad (2.49)$$

$$i[P^\mu, M^{\rho\sigma}] = g^{\mu\rho} P^\sigma - g^{\mu\sigma} P^\rho, \quad (2.50)$$

$$[P^\mu, P^\nu] = 0. \quad (2.51)$$

These relations can be derived from

$$U(\Lambda, a)U(\Lambda', a')U^{-1}(\Lambda, a) = U(\Lambda\Lambda'\Lambda^{-1}, a + \Lambda a' - \Lambda\Lambda'\Lambda^{-1}a), \quad (2.52)$$

which follows from the composition rules for the $T(\Lambda, a)$: insert infinitesimal transformations (2.48) for each $U(\Lambda = 1 + \varepsilon, a)$, with $U^{-1}(\Lambda, a) = U(1 - \varepsilon, -a)$, keep only linear terms in all group parameters $\varepsilon, \varepsilon', a$ and a' , and compare coefficients of the terms $\sim \varepsilon\varepsilon', a\varepsilon', \varepsilon a'$ and aa' . A shortcut to arrive at the Lorentz algebra relation (2.49) is to calculate the generator $M^{\mu\nu}$ directly in the four-dimensional representation, where $U(\Lambda, 0) = \Lambda$ is the Lorentz transformation itself:

$$U(\Lambda, 0)^\alpha_\beta = \delta^\alpha_\beta + \frac{i}{2}\varepsilon_{\mu\nu} (M^{\mu\nu})^\alpha_\beta + \dots = \Lambda^\alpha_\beta = \delta^\alpha_\beta + \varepsilon^\alpha_\beta + \dots \quad (2.53)$$

This is solved by the tensor

$$(M^{\mu\nu})^\alpha_\beta = -i(g^{\mu\alpha}\delta^\nu_\beta - g^{\nu\alpha}\delta^\mu_\beta) \quad (2.54)$$

which satisfies the commutator relation (2.49).

We can cast the Poincaré algebra relations in a less compact but more useful form. The antisymmetric matrix $\varepsilon_{\mu\nu}$ contains the six group parameters and the antisymmetric matrix $M^{\mu\nu}$ the six generators. If we define the generator of $SO(3)$ rotations \mathbf{J} (the angular momentum) and the generator of boosts \mathbf{K} via

$$M^{ij} = -\varepsilon_{ijk} J^k \quad \Leftrightarrow \quad J^i = -\frac{1}{2}\varepsilon_{ijk} M^{jk}, \quad M^{0i} = K^i, \quad (2.55)$$

(Ex)

then the commutator relations take the form

$$\begin{aligned}
[J^i, J^j] &= i\varepsilon_{ijk} J^k, & [J^i, P^j] &= i\varepsilon_{ijk} P^k, & [P^i, P^j] &= 0, \\
[J^i, K^j] &= i\varepsilon_{ijk} K^k, & [K^i, P^j] &= i\delta_{ij} P_0, & [J^i, P_0] &= 0, \\
[K^i, K^j] &= -i\varepsilon_{ijk} J^k, & [K^i, P_0] &= iP^i, & [P^i, P_0] &= 0.
\end{aligned}
\tag{2.56}$$

Here we see that boosts and rotations generally do not commute unless the boost and rotation axes coincide. Moreover, P_0 (which is the Hamilton operator in the quantum field theory) commutes with rotations and spatial translations but not with boosts and therefore the eigenvalues of \mathbf{K} cannot be used for labeling physical states. If we similarly define $\varepsilon_{ij} = -\varepsilon_{ijk} \phi^k$ and $\varepsilon_{0i} = s^i$, we obtain

$$\frac{i}{2} \varepsilon_{\mu\nu} M^{\mu\nu} = i\boldsymbol{\phi} \cdot \mathbf{J} + i\mathbf{s} \cdot \mathbf{K} . \tag{2.57}$$

\mathbf{J} is hermitian but \mathbf{K} is antihermitian for all finite-dimensional representations, which prevents them from being unitary: there are no finite-dimensional unitary representations of the Lorentz and Poincaré groups. This is a consequence of the fact that the Lorentz group is not compact: it contains the boosts whose parameter space is isomorphic to \mathbb{R}^3 . Later will discuss explicit examples for \mathbf{K} when considering spinor representations.

Representation on the Fock space. How is the Poincaré group represented on the Fock space? It is not an accident that we chose the same symbol P^μ for the generator of translations *and* for the classical Noether charge in Eq. (1.51), which meanwhile has also become the momentum operator in the quantum field theory. It turns out that, after quantizing the theory, the classical constants of motion P^μ and $M^{\mu\nu}$ become self-adjoint operators on the Fock space which define a unitary representation of the Poincaré group.² This means they satisfy the same Poincaré algebra relations as in Eqs. (2.49–2.51), which happens to be a consequence of the commutation relations for the fields, and the corresponding operator $U(\Lambda, a)$ is unitary.

In Eqs. (1.31–1.32) we have seen how classical fields behave under Poincaré transformations. The general transformation behavior of a collection of field *operators* $\Phi_i(x)$ under Poincaré transformations is imposed as an *axiom* of quantum field theory:

$$U(\Lambda, a) \Phi_i(x) U(\Lambda, a)^{-1} = D(\Lambda)_{ij}^{-1} \Phi_j(\Lambda x + a) . \tag{2.58}$$

It ensures that matrix elements of field operators transform as

$$\langle \lambda'_1 | \Phi_i(x') | \lambda'_2 \rangle = \langle \lambda_1 | U(\Lambda, a)^{-1} \Phi_i(x') U(\Lambda, a) | \lambda_2 \rangle \stackrel{!}{=} D(\Lambda)_{ij} \langle \lambda_1 | \Phi_j(x) | \lambda_2 \rangle . \tag{2.59}$$

This can be generalized to products of field operators at different spacetime points, which gives the transformation behavior of correlation functions. We will discuss the consequences of Eq. (2.58) in more detail later in the context of the Dirac field. For the moment we restrict ourselves to a single scalar field where the equation reduces to

$$U(\Lambda, a) \Phi(x) U(\Lambda, a)^{-1} = \Phi(\Lambda x + a) . \tag{2.60}$$

²Unitarity of $U(\Lambda, a)$ has now become possible because the Fock space is infinite-dimensional.

In particular, for translations $U(1, a) = e^{ia_\mu P^\mu}$ it takes the form

$$e^{ia_\mu P^\mu} \Phi(x) e^{-ia_\mu P^\mu} = \Phi(x + a). \quad (2.61)$$

Expanding both sides of the equation to $\mathcal{O}(a)$ we obtain

$$\Phi(x) + ia_\mu [P^\mu, \Phi(x)] + \dots = \Phi(x) + a_\mu \partial^\mu \Phi(x) + \dots \quad (2.62)$$

from where we obtain the **Heisenberg equations of motion**:

$$\partial_\mu \Phi(x) = i[P_\mu, \Phi(x)]. \quad (2.63)$$

Since they follow from translation invariance they are quite general: they do not only hold for scalar fields but also for polynomials in Φ , and more generally also for the individual components of fields with higher spin because each component behaves like a scalar field under translations. In particular, we can read off the Heisenberg equation for the time evolution which is known from quantum mechanics:

$$\frac{\partial \Phi(x)}{\partial t} = i[H, \Phi(x)]. \quad (2.64)$$

From the Heisenberg equations for $\Phi(x)$ and $\Pi(x)$ one can further recover the Klein-Gordon equation for the field $\Phi(x)$.

The analogue of Eq. (2.61) derived from Lorentz invariance has the form

$$e^{\frac{i}{2}\varepsilon_{\mu\nu} M^{\mu\nu}} \Phi(x) e^{-\frac{i}{2}\varepsilon_{\mu\nu} M^{\mu\nu}} = \Phi(\Lambda x), \quad (2.65)$$

with $\Lambda = 1 + \varepsilon + \dots$. Expanding both sides to $\mathcal{O}(\varepsilon)$ and exploiting the antisymmetry of $\varepsilon_{\mu\nu}$ yields the equation

$$i(x_\mu \partial_\nu - x_\nu \partial_\mu) \Phi(x) = [M_{\mu\nu}, \Phi(x)]. \quad (2.66)$$

Causality. The basic postulate of quantum field theory is that two measurements at spacelike distances should not affect each other. This is guaranteed if any two local observables $\mathcal{O}_1(x)$ and $\mathcal{O}_2(y)$ at spacelike separation commute, i.e.,

$$[\mathcal{O}_1(x), \mathcal{O}_2(y)] \stackrel{!}{=} 0 \quad \text{if } (x - y)^2 < 0. \quad (2.67)$$

To this end, consider the commutator of two fields at arbitrary times:

$$\Delta(x - y) := [\Phi(x), \Phi(y)]. \quad (2.68)$$

This quantity is known by various names: Pauli-Jordan function, Schwinger's Δ function, or simply causal propagator. If we insert the Fourier decomposition (2.3) for free fields, use the commutator relation (2.5) and set $z = x - y$, we immediately get

$$\Delta(z) = \frac{1}{(2\pi)^3} \int \frac{d^3 p}{2E_p} (e^{-ipz} - e^{ipz}) \Big|_{p^0=E_p}. \quad (2.69)$$

Now observe this: for $z^0 = 0$, $\Delta(z)$ vanishes because in that case we can change the integration variable from $\mathbf{p} \rightarrow -\mathbf{p}$ and the difference cancels. On the other hand, $\Delta(z)$ is Lorentz-invariant because both $e^{\pm i\mathbf{p}z}$ and the integral measure $d^3p/(2E_p)$ are Lorentz-invariant. A Lorentz-invariant quantity that vanishes for $z^0 = 0$ must vanish for all spacelike z with $z^2 < 0$, because they can all be reached by a Lorentz transformation. Hence, $\Delta(z)$ has only support inside the light cone ($z^2 \geq 0$).

In an interacting quantum field theory we cannot use a free mode expansion anymore to calculate $\Delta(z)$. In that case we also have to postulate **microcausality** as an axiom:

$$[\Phi(x), \Phi(y)] = 0 \quad \text{if } (x - y)^2 < 0. \quad (2.70)$$

This also generalizes our earlier commutation relations (2.2) because they can be derived from it: $\partial_0 \Delta(z)|_{z^0=0} = -i\delta^3(\mathbf{z})$. Of course $\Phi(x)$ is not a measurable quantity but actual observables like currents, charges etc. are functions of the fields and therefore inherit its causal properties.

Propagators. Consider now the quantity

$$D(x - y) := \langle 0 | \Phi(x) \Phi(y) | 0 \rangle. \quad (2.71)$$

Since $\Phi(x) | 0 \rangle = | x \rangle$, this is the amplitude $\langle x | y \rangle$ for a particle that is emitted at y and propagates to x . Its analogue in nonrelativistic quantum mechanics is the amplitude $\langle \mathbf{x} | e^{-iHt} | \mathbf{y} \rangle$, which is nonzero even if $x - y$ is spacelike (hence the problem with causality in quantum mechanics).³ If we insert the Fourier decomposition (2.3) into $D(x - y)$ then, because we act on the vacuum on both sides, the only term that survives is $\langle 0 | a_p a_p^\dagger | 0 \rangle = \langle p | p' \rangle = 2E_p \delta^3(\mathbf{p} - \mathbf{p}')$ and we arrive at

$$D(z) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_p} e^{-ipz} \Big|_{p^0=E_p}. \quad (2.72)$$

This expression is again Lorentz-invariant but nonzero for $z^0 = 0$, so it is generally also nonzero for spacelike distances $z^2 < 0$. How is this compatible with causality? The crucial observation is that the commutator

$$[\Phi(x), \Phi(y)] = D(x - y) - D(y - x) \quad (2.73)$$

describes two physical processes (propagation from $y \rightarrow x$ and $x \rightarrow y$) whose amplitudes cancel each other for $(x - y)^2 < 0$. This makes indeed sense because both processes can occur: if $x - y$ is spacelike, there is no Lorentz-invariant notion of whether $x^0 - y^0$ is larger or smaller than zero.

We can understand this better if we consider a *complex* scalar field where the Fourier decomposition of the field $\Phi(x)$ contains a_p , b_p^\dagger and $\Phi^\dagger(x) \sim b_p$, a_p^\dagger . In that case we have to modify the axiom (2.70) so that it becomes

$$\Delta(x - y) = [\Phi(x), \Phi^\dagger(y)] = 0 \quad \text{if } (x - y)^2 < 0, \quad (2.74)$$

³We do not need to insert the time-evolution operator in $D(x - y)$ because $|x\rangle$ already contains information about the time variable x^0 .

whereas $[\Phi(x), \Phi(y)] = 0 \forall x, y$. The result for $\Delta(z)$ in Eq. (2.69) remains the same, but now we have

$$[\Phi(x), \Phi^\dagger(y)] = \langle 0 | \Phi(x) \Phi^\dagger(y) | 0 \rangle - \langle 0 | \Phi^\dagger(y) \Phi(x) | 0 \rangle. \quad (2.75)$$

The first term corresponds to a particle that travels from $y \rightarrow x$ and the second term to an *antiparticle* travelling from $x \rightarrow y$, and both processes cancel each other in the commutator. Therefore, it is really the multiparticle nature of quantum field theory that saves causality: the particle and antiparticle propagation cancel each other. (For a real scalar field $\Phi(x) = \Phi^\dagger(x)$ the particle is its own antiparticle.)

Feynman propagator. Taking this idea further, we define the Feynman propagator

$$D_F(x - y) := \langle 0 | \mathbb{T} \Phi(x) \Phi^\dagger(y) | 0 \rangle = \begin{cases} \langle 0 | \Phi(x) \Phi^\dagger(y) | 0 \rangle & \text{if } x^0 \geq y^0, \\ \langle 0 | \Phi^\dagger(y) \Phi(x) | 0 \rangle & \text{if } y^0 \geq x^0, \end{cases} \quad (2.76)$$

where the time-ordering \mathbb{T} of some product of field operators implies that they should be ordered with increasing times from right to left. The Feynman propagator will become extremely important later because it is the fundamental quantity that appears in the Feynman rules for S -matrix elements. It describes the propagation of a particle forward in time, but simultaneously also the propagation of an antiparticle ‘backward in time’; hence, these two processes are physically the same.

The various propagators that we encountered are also called **Green’s functions** because they are the Green functions of the Klein-Gordon equation:

$$(\square + m^2) iD(z) = \delta^4(z). \quad (2.77)$$

We can find the general solution to this equation by taking the Fourier transform of both sides:

$$D(z) = \frac{1}{(2\pi)^4} \int d^4p \tilde{D}(p) e^{-ipz}, \quad \delta^4(z) = \frac{1}{(2\pi)^4} \int d^4p e^{-ipz}, \quad (2.78)$$

so that the propagator in momentum space becomes

$$\tilde{D}(p) = \frac{i}{p^2 - m^2}. \quad (2.79)$$

It has a pole on the real axis of $p^2 = m^2$ or, equivalently, two poles at positive and negative energies $p^0 = \pm E_p = \pm \sqrt{\mathbf{p}^2 + m^2}$. The strategy in order to return the propagator to real space is to carry out the p^0 integration first:

$$D(z) = i \int \frac{d^3p}{(2\pi)^4} e^{i\mathbf{p}\cdot\mathbf{z}} \int dp_0 \frac{e^{-ip_0 z_0}}{p_0^2 - E_p^2}. \quad (2.80)$$

The integral can be calculated by extending the real-axis integration to a closed contour in the complex p_0 plane, which is allowed as long as the integrand vanishes at complex infinity. For $z^0 > 0$ this holds as long as we close the contour in the lower half plane, whereas for $z^0 < 0$ the integrand at infinity only vanishes if we close it in the upper half plane. Any consistent prescription to avoid divergences in performing this contour

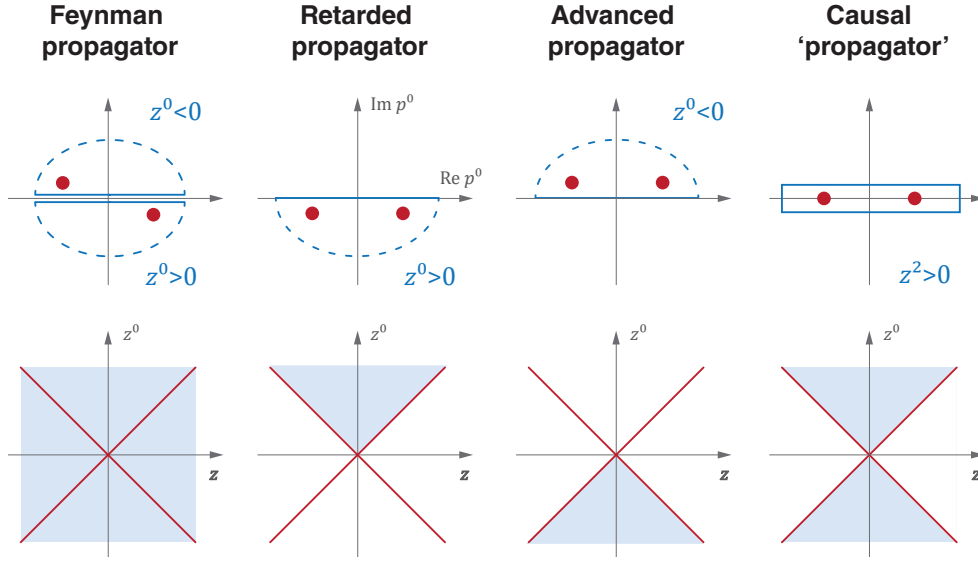


FIGURE 2.1: Various integration contours in the complex p^0 plane (top row) and support of the resulting propagators in the Minkowski diagram $(z^0, |z|)$ (bottom row).

integral (there are 2×2 different ways of doing so) leads to a solution of the original equation (2.77). By the residue theorem

$$\oint dz f(z) = 2\pi i \sum_n R(z_n), \quad R(z_0) = \lim_{z \rightarrow z_0} (z - z_0) f(z) \quad (2.81)$$

the result is $(2\pi i)$ times the sum of the residues at $p^0 = \pm E_p$, which are given by

$$R_+ = \frac{e^{-iE_p z_0}}{2E_p}, \quad R_- = -\frac{e^{iE_p z_0}}{2E_p}. \quad (2.82)$$

To arrive at the Feynman propagator, we must integrate slightly below and above the p^0 axis for $\text{Re } p^0 < 0$ and $\text{Re } p^0 > 0$, respectively (see Fig. 2.1). For $z_0 > 0$, we close the contour in the lower half plane (because the integral at infinity vanishes only below) and pick up the positive energy pole. For $z_0 < 0$, we close the contour in the upper half plane and pick up the negative energy pole, so the p^0 integral becomes

$$\int dp_0 \frac{e^{-ip_0 z_0}}{p_0^2 - E_p^2} = 2\pi i [-\Theta(z_0) R_+ + \Theta(-z_0) R_-], \quad (2.83)$$

where the positive residue comes with a minus because of the opposite integration direction. In total, the Feynman propagator ‘propagates positive energies forward in time and negative energies backwards’:

$$\begin{aligned} D_F(z) &= \int \frac{d^3 p}{(2\pi)^4} e^{i\mathbf{p} \cdot \mathbf{z}} i(2\pi i) [-\Theta(z_0) R_+ + \Theta(-z_0) R_-] \\ &= \int \frac{d^3 p}{2E_p} \frac{\Theta(z_0) e^{-ipz} + \Theta(-z_0) e^{ipz}}{(2\pi)^3} \Big|_{p^0=E_p} \\ &\stackrel{(2.72)}{=} \Theta(z^0) D(z) + \Theta(-z^0) D(-z), \end{aligned} \quad (2.84)$$

and we see that this is indeed the definition of the Feynman propagator in Eq. (2.76). Note that instead of deforming the integration path in p^0 we could have equally shifted the poles by $\pm E_p \rightarrow \pm E'_p = \pm(E_p - i\epsilon/(2E_p))$, as indicated in Fig. 2.1:

$$\frac{1}{p_0^2 - E_p^2} \rightarrow \frac{1}{p_0^2 - E'_p{}^2} = \frac{1}{p_0^2 - E_p^2 + i\epsilon} = \frac{1}{p^2 - m^2 + i\epsilon}, \quad (2.85)$$

so we can equivalently write the Feynman propagator as

$$D_F(z) = \int \frac{d^4 p}{(2\pi)^4} e^{-ipz} \frac{i}{p^2 - m^2 + i\epsilon}. \quad (2.86)$$

Later we will see that this ‘ $i\epsilon$ prescription’ follows from the imaginary-time boundary conditions when projecting Green functions onto the interacting vacuum, which is why it is really the *Feynman* propagator that appears in the interacting quantum field theory (and not any of the other options in performing the contour integral).

Retarded and advanced propagators. For completeness we discuss two other physically relevant integration paths. One is to integrate slightly above *both* poles and the other is to integrate slightly below them (see Fig. 2.1). In the first case, for $z^0 > 0$ we must close the contour in the lower half plane (which gives the sum of the residues) and for $z^0 < 0$ in the upper half plane (which gives zero); the situation is reversed in the second case. The resulting propagators are the *retarded* and *advanced* propagators:

$$\begin{aligned} D_{R,A}(z) &= \mp \int \frac{d^3 p}{(2\pi)^4} e^{i\mathbf{p}\cdot\mathbf{z}} i(2\pi i) \Theta(\pm z^0) (R_+ + R_-) \\ &= \pm \Theta(\pm z^0) \int \frac{d^3 p}{2E_p} \frac{e^{-ipz} - e^{ipz}}{(2\pi)^3} \Big|_{p^0=E_p} \\ &= \pm \Theta(\pm z^0) (D(z) - D(-z)) = \pm \Theta(\pm z^0) \Delta(z). \end{aligned} \quad (2.87)$$

The retarded propagator has only support in the forward light cone and the advanced propagator in the backward light cone. They also appear in classical field theory in the context of constructing solutions to the inhomogeneous Klein-Gordon equation, where they propagate the inhomogeneity forward (D_R) and backward (D_A) in time. The classical version of causality states that D_R and D_A vanish if $(x-y)^2 < 0$, which we also proved here. By contrast, the Feynman propagator D_F has no classical counterpart. It does not vanish for spacelike distances but rather falls off exponentially outside the light cone.

3 Dirac field

Earlier in Eqs. (1.31–1.32) we claimed that, under Poincaré transformations $x' = \Lambda x + a$, a generic set of *classical* fields $\Phi_i(x)$ transforms as

$$\Phi'_i(x') = D(\Lambda)_{ij} \Phi_j(x), \quad (3.1)$$

and the quantum version of this relation for field *operators* was given in Eq. (2.58):

$$U(\Lambda, a) \Phi_i(x) U(\Lambda, a)^{-1} = D(\Lambda)_{ij}^{-1} \Phi_j(\Lambda x + a). \quad (3.2)$$

We have already worked out the structure of $U(\Lambda, a)$ (at least a little bit): it contains the generators P^μ of translations and $M^{\mu\nu}$ of Lorentz transformations, which are now understood as operators on the Fock space. For example, we established the momentum operator for a free scalar theory in Eq. (2.22), and it is easy to show that it satisfies indeed the Lie algebra relation $[P^\mu, P^\nu] = 0$.

Irreducible representations of the Lorentz group. The missing link in both cases is the matrix $D(\Lambda)$. Because it refers to the indices i and j in the equations above, it classifies which types of fields can actually appear in a Lagrangian: scalar, Dirac, vector fields etc. We will see that $D(\Lambda)$ also provides the **spin** contribution to observables. For scalar fields $D(\Lambda) = 1$ and so we could simply ignore it. In general, $D(\Lambda)$ is a finite-dimensional irreducible representation matrix of the Lorentz group, so it must share the same structure with $U(\Lambda, 0)$:

$$D(\Lambda) = e^{\frac{i}{2} \varepsilon_{\mu\nu} M^{\mu\nu}} = e^{i\phi \cdot \mathbf{J} + i\mathbf{s} \cdot \mathbf{K}}, \quad M^{ij} = -\varepsilon_{ijk} J^k, \quad M^{0i} = K^i. \quad (3.3)$$

That is, in an n -dimensional representation $D(\Lambda)$, $M^{\mu\nu}$, \mathbf{J} and \mathbf{K} are $n \times n$ matrices. Of course $M^{\mu\nu}$ is not the same as the Fock-space operator that was just mentioned before, but let's keep the generic notation for the moment to avoid clutter. What do these matrices look like? Can they have any dimensionality?

Let's build a Lorentz tensor of rank n . It is defined by the transformation law

$$(T')^{\mu\nu\dots\tau} = \underbrace{\Lambda^\mu_\alpha \Lambda^\nu_\beta \dots \Lambda^\tau_\lambda}_{n \text{ times}} T^{\alpha\beta\dots\lambda}, \quad (3.4)$$

so we can always construct the representation matrices $\Lambda^\mu_\alpha \Lambda^\nu_\beta \dots$ of the Lorentz transformation as the outer product $\mathbf{4} \otimes \mathbf{4} \otimes \dots$ of the 4-dimensional defining representation Λ . However, these representations are not irreducible. Take for example the 4×4 tensor $T^{\mu\nu}$, which has in principle 16 components. Its trace, its antisymmetric component, and its symmetric and traceless part,

$$S = T^\alpha_\alpha, \quad A^{\mu\nu} = \frac{1}{2} (T^{\mu\nu} - T^{\nu\mu}), \quad S^{\mu\nu} = \frac{1}{2} (T^{\mu\nu} + T^{\nu\mu}) - \frac{1}{4} g^{\mu\nu} S, \quad (3.5)$$

do not mix under Lorentz transformations: an (anti-)symmetric tensor is still (anti-)symmetric after the transformation, and the trace S is Lorentz-invariant. The trace is one-dimensional, the antisymmetric part defines a 6-dimensional subspace, and the

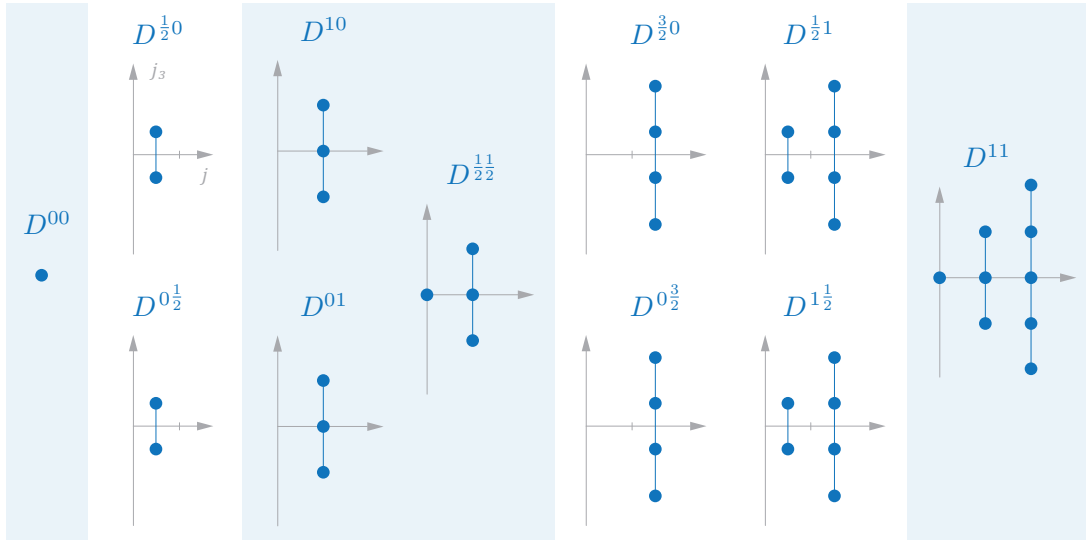


FIGURE 3.1: Multiplets of the Lorentz group: tensor (shaded) vs. spinor representations. The number of states in a multiplet gives the dimension of the representation.

symmetric and traceless part a 9-dimensional subspace. Therefore, we have the decomposition $\mathbf{4} \otimes \mathbf{4} = \mathbf{1} \oplus \mathbf{6} \oplus \mathbf{9}$, which means there must be at least representations with dimensions 1, 4, 6 and 9. How many more are there?

There is a simple way to classify the irreducible representations of the Lorentz group. If we define

$$\mathbf{A} = \frac{1}{2}(\mathbf{J} - i\mathbf{K}), \quad \mathbf{B} = \frac{1}{2}(\mathbf{J} + i\mathbf{K}) \quad (3.6)$$

and calculate their commutator relations using Eq. (2.56), we obtain two copies of an $SU(2)$ algebra with hermitian generators A_i and B_i :

$$[A_i, A_j] = i\varepsilon_{ijk} A_k, \quad [B_i, B_j] = i\varepsilon_{ijk} B_k, \quad [A_i, B_j] = 0. \quad (3.7)$$

We are familiar with $SU(2)$: the two Casimir operators \mathbf{A}^2 and \mathbf{B}^2 have eigenvalues $a(a+1)$ and $b(b+1)$, hence there are two quantum numbers $a, b = 0, \frac{1}{2}, 1, \dots$ to label the multiplets. We denote the irreducible representations by D^{ab} ; their dimension must be $(2a+1)(2b+1)$. The generator of rotations is $\mathbf{J} = \mathbf{A} + \mathbf{B}$, so we can use the $SU(2)$ angular momentum addition rules to construct the states within each multiplet: the states come with all possible spins $j = |a-b| \dots a+b$, where j_3 goes from $-j$ to j . The multiplets are visualized in Fig. 3.1.

The ‘tensor representations’, where $a+b$ is integer (the shaded multiplets in Fig. 3.1), are the actual irreducible representations of the Lorentz group that can be constructed via Eq. (3.4):

- **Lorentz scalars** transform under the trivial representation D^{00} , where the generator is $M^{\mu\nu} = 0$ and the representation matrix is $D(\Lambda) = 1$.

- A **Lorentz vector** transforms under the four-dimensional vector representation $D^{\frac{1}{2}\frac{1}{2}}$. It plays a special role because the transformation matrix is $D(\Lambda) = \Lambda$ itself, and it can be used to construct all further (reducible) tensor representations according to Eq. (3.4). The generator $M^{\mu\nu}$ has the form of Eq. (2.54).
- A **symmetric and traceless tensor** $S^{\mu\nu}$ transforms under the 9-dimensional ‘tensor’ representation D^{11} .
- An **antisymmetric tensor** $A^{\mu\nu}$ transforms under the six-dimensional antisymmetric representation. If $A^{\mu\nu}$ is real it is also irreducible; if it is complex (which it is in Euclidean space) it can be further decomposed into a self-dual (D^{10}) and an anti-self-dual representation (D^{01}), depending on the sign of the condition $A^{\mu\nu} = \pm \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} A_{\rho\sigma}$.

These are the representations **1**, **4**, **6** and **9** that we anticipated above. However, what is more interesting in view of Dirac fields are the **spinor representations** where $a + b$ is half-integer. They are not representations of the Lorentz group but rather of the group $SL(2, \mathbb{C})$, which is the set of complex 2×2 matrices with unit determinant. Like the Lorentz group, it also depends on six real parameters and it has the same Lie algebra. From the point of view of the Lorentz group, the spinor representations are merely *projective* representations, where instead of $D(\Lambda') D(\Lambda) = D(\Lambda' \Lambda)$ one has $D(\Lambda') D(\Lambda) = \pm D(\Lambda' \Lambda)$, so they are double-valued. However, both of them are physically equivalent and therefore the representations in Fig. 3.1 are all relevant.

The origin of this behavior is the rotational subgroup $SO(3)$ of the Lorentz group which is not simply connected. The projective representations of a group correspond to the representations of its universal covering group: it has the same Lie algebra, which reflects the property of the group close to the identity, but it is simply connected. In the same way as $SU(2)$ is the double cover of $SO(3)$, the double cover of $SO(3, 1)^\dagger$ is the group $SL(2, \mathbb{C})$. A double-valued projective representation of $SO(3, 1)^\dagger$ corresponds to a single-valued representation of $SL(2, \mathbb{C})$. Similarly, the double cover of the *Euclidean* Lorentz group $SO(4)$ is $SU(2) \times SU(2)$; these are the representations that we actually derived in Fig. 3.1.

The fundamental spinor representations are $D^{\frac{1}{2}0}$ and $D^{0\frac{1}{2}}$ because all other ones can be built from them. They have both dimension two and carry spin $j = 1/2$. Because one of the Casimir eigenvalues a or b is zero, we say that they have definite *chirality*: $D^{\frac{1}{2}0}$ is the left-handed and $D^{0\frac{1}{2}}$ the right-handed representation. We can immediately write down 2×2 matrices that satisfy the $SU(2)$ Lie algebra, namely the Pauli matrices:

$$\left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2} \right] = i \varepsilon_{ijk} \frac{\sigma^k}{2}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.8)$$

Therefore the generators and transformation matrices are

$$\begin{aligned} D^{\frac{1}{2}0}: \quad \mathbf{A} = \frac{\boldsymbol{\sigma}}{2}, \quad \mathbf{B} = 0 & \quad \Rightarrow \quad \mathbf{J} = \frac{\boldsymbol{\sigma}}{2}, \quad \mathbf{K} = i \frac{\boldsymbol{\sigma}}{2} & \quad \Rightarrow \quad D_L(\Lambda) = e^{i\phi \cdot \frac{\boldsymbol{\sigma}}{2} - \mathbf{s} \cdot \frac{\boldsymbol{\sigma}}{2}}, \\ D^{0\frac{1}{2}}: \quad \mathbf{A} = 0, \quad \mathbf{B} = \frac{\boldsymbol{\sigma}}{2} & \quad \Rightarrow \quad \mathbf{J} = \frac{\boldsymbol{\sigma}}{2}, \quad \mathbf{K} = -i \frac{\boldsymbol{\sigma}}{2} & \quad \Rightarrow \quad D_R(\Lambda) = e^{i\phi \cdot \frac{\boldsymbol{\sigma}}{2} + \mathbf{s} \cdot \frac{\boldsymbol{\sigma}}{2}}. \end{aligned} \quad (3.9)$$

The representation matrices $D_{L,R}(\Lambda) \in SL(2, \mathbb{C})$ are complex 2×2 matrices, and the corresponding spinors are left- and right-handed **Weyl spinors** ψ_L, ψ_R that transform as

$$\psi'_L(x') = D_L(\Lambda) \psi_L(x), \quad \psi'_R(x') = D_R(\Lambda) \psi_R(x). \quad (3.10)$$

We can check that these are only projective representations. Consider, for example, a rotation by $\phi = 2\pi$ around the z -axis: the Lorentz transformation is $\Lambda = 1$, but the representation matrices become $D_{L,R}(1) = e^{i\pi\sigma_3} = \cos \pi + i\sigma_3 \sin \pi = -1$, and only a rotation by 4π will bring them back to 1.

In principle, the Weyl representation would be sufficient to describe spin- $\frac{1}{2}$ fields. However, the problem is that under a parity transformation the rotation generators are invariant whereas the boost generators change their sign: $\mathbf{J} \rightarrow \mathbf{J}$, $\mathbf{K} \rightarrow -\mathbf{K}$. Therefore, parity exchanges $\mathbf{A} \leftrightarrow \mathbf{B}$ in Eq. (3.6) and transforms the two fundamental representations into each other. A theory that is invariant under parity (such as QED and QCD, but not the weak interaction) must necessarily include both doublets, because we cannot write down a parity-invariant Lagrangian with ψ_L or ψ_R alone. In such a combined Lagrangian the dynamics will couple ψ_L and ψ_R together. This is a consequence of Eq. (3.9) because $D_L(\Lambda)^\dagger = D_R(\Lambda)^{-1}$, and a Lorentz-invariant Lagrangian will contain terms $\sim \psi_L^\dagger \psi_R$, $\psi_R^\dagger \psi_L$ that are separately Lorentz-invariant.

Instead of carrying around the left- and right-handed Weyl spinors, it is more convenient to combine them into **Dirac spinors** ψ_α with $\alpha = 1 \dots 4$. They can be constructed as the direct sums of ψ_L and ψ_R , hence we denote the (reducible) Dirac representation by $D^{\frac{1}{2}0} \oplus D^{0\frac{1}{2}}$:

$$\mathbf{J} = \begin{pmatrix} \boldsymbol{\sigma}/2 & 0 \\ 0 & \boldsymbol{\sigma}/2 \end{pmatrix} =: \frac{\boldsymbol{\Sigma}}{2}, \quad \mathbf{K} = \begin{pmatrix} i\boldsymbol{\sigma}/2 & 0 \\ 0 & -i\boldsymbol{\sigma}/2 \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}. \quad (3.11)$$

The resulting generator $M^{\mu\nu}$ constructed via Eq. (3.3) is consequently a 4×4 matrix that satisfies again the Lorentz algebra relation (2.49). It leads to a four-dimensional transformation matrix

$$D(\Lambda) = e^{\frac{i}{2} \varepsilon_{\mu\nu} M^{\mu\nu}} = e^{i\boldsymbol{\phi} \cdot \mathbf{J} + i\mathbf{s} \cdot \mathbf{K}} \quad (3.12)$$

which transforms the spinors as $\psi'_\alpha(x') = D(\Lambda)_{\alpha\beta} \psi_\beta(x)$.

Clifford algebra. It is still desirable to have a manifestly covariant notation. This is where the Clifford algebra comes in: it is the algebra spanned by the $n \times n$ matrices γ^μ , with $\mu = 0 \dots 3$, so that the anticommutator is

$$\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \mathbf{1}_{n \times n}. \quad (3.13)$$

This implies $(\gamma^0)^2 = 1$, $(\gamma^i)^2 = -1$ and $\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu$ for $\mu \neq \nu$. The Clifford algebra is quite useful because every representation of it *induces* a representation of the Lorentz algebra via the definition

$$M^{\mu\nu} := -\frac{i}{4} [\gamma^\mu, \gamma^\nu]. \quad (3.14)$$

That is, by using the anticommutator relation (3.13) one can show that $M^{\mu\nu}$ satisfies the Lorentz algebra relation (2.49). Consequently, for $n = 4$ there must be an explicit form for the γ -matrices where $M^{\mu\nu}$ reproduces Eq. (3.11); it is called the *chiral* or *Dirac* representation:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \Leftrightarrow \gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad (3.15)$$

(Ex)

where we abbreviated $\sigma^\mu = (\mathbf{1}, \sigma^i)$ and $\bar{\sigma}^\mu = (\mathbf{1}, -\sigma^i)$. We also define

$$\gamma^5 = \gamma_5 := \frac{i}{4!} \varepsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \stackrel{\text{chiral rep.}}{=} \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \quad (3.16)$$

with the properties $(\gamma_5)^\dagger = \gamma_5$, $(\gamma_5)^2 = 1$ and $\{\gamma^\mu, \gamma_5\} = 0$. The totally antisymmetric tensor $\varepsilon_{\mu\nu\rho\sigma}$ is defined as

$$\varepsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of } 0123 \\ -1 & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of } 0123 \\ 0 & \text{otherwise.} \end{cases} \quad (3.17)$$

It switches sign if *spatial* indices are raised or lowered, which entails $\varepsilon_{\mu\nu\rho\sigma} = -\varepsilon^{\mu\nu\rho\sigma}$. The matrix γ_5 is useful for constructing the *chiral projectors* $(1 \pm \gamma_5)/2$ onto the Weyl spinors:

$$\frac{1 - \gamma_5}{2} \psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}, \quad \frac{1 + \gamma_5}{2} \psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}. \quad (3.18)$$

The chiral representation is where the group structure is most transparent because the generators \mathbf{J} and \mathbf{K} are the direct sums of the two-dimensional matrices. Expressed in terms of gamma matrices they are given by $\mathbf{\Sigma} = \gamma_5 \gamma^0 \boldsymbol{\gamma}$ and $\mathbf{K} = -\frac{i}{2} \gamma^0 \boldsymbol{\gamma}$, which follows from Eqs. (3.3), (3.14) and (3.16). It is also practical for calculations in the ultrarelativistic limit where masses can be neglected.

It follows from Eq. (3.13) that with every invertible matrix U also $U\gamma^\mu U^{-1}$ is a representation of the Clifford algebra, and $U\psi$ is the spinor in the new representation. For example, the *standard* representation

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \quad (3.19)$$

is frequently used because it is convenient for calculations in the non-relativistic limit. It emerges from the chiral representation through the matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ -\mathbf{1} & \mathbf{1} \end{pmatrix} \Rightarrow \psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_R + \psi_L \\ \psi_R - \psi_L \end{pmatrix} =: \begin{pmatrix} \phi \\ \chi \end{pmatrix}. \quad (3.20)$$

By multiplying the γ -matrices with each other one can form a complete system of 4×4 matrices, which consists of 16 matrices $\Gamma_1 \dots \Gamma_{16}$:

$$\mathbf{1}, \quad \gamma^\mu, \quad \sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu], \quad \gamma^\mu \gamma^5, \quad \gamma^5. \quad (3.21)$$

They are orthonormal with respect to the scalar product $\frac{1}{4} \text{Tr}(\Gamma_i^\dagger \Gamma_j) = \delta_{ij}$ and (except for Γ_1) traceless: $\text{Tr} \Gamma_i = \delta_{i1}$. Therefore we can express any 4×4 matrix by

$$A = \sum_{i=1}^{16} c_i \Gamma_i, \quad c_i = \frac{1}{4} \text{Tr}(\Gamma_i^\dagger A). \quad (3.22)$$

Lorentz bilinears. How can we construct Lorentz invariants from a spinor $\psi(x)$? We already know that under Lorentz transformations we have $\psi'(x') = D(\Lambda)\psi(x)$. Let's try the combination

$$\psi^\dagger(x)\psi(x) \quad \rightarrow \quad \psi'^\dagger(x')\psi'(x') = \psi^\dagger(x)D(\Lambda)^\dagger D(\Lambda)\psi(x) \quad (3.23)$$

For $D(\Lambda)^\dagger = D(\Lambda)^{-1}$ this would be a Lorentz scalar. However, $D(\Lambda)$ cannot be unitary because it contains the boosts: \mathbf{J} is hermitian but \mathbf{K} is antihermitian, and consequently $M^{\mu\nu}$ cannot be hermitian:

$$M_{\mu\nu}^\dagger = \left(-\frac{i}{4} [\gamma^\mu, \gamma^\nu] \right)^\dagger = -\frac{i}{4} [\gamma^{\mu\dagger}, \gamma^{\nu\dagger}] \neq -\frac{i}{4} [\gamma^\mu, \gamma^\nu]. \quad (3.24)$$

From the point of view of the Clifford algebra, it is impossible to make all γ -matrices hermitian: since $(\gamma^0)^2 = 1$, γ^0 has real eigenvalues, but $(\gamma^i)^2 = -1$ and therefore the eigenvalues of γ^i are imaginary. What we can write instead is

$$(\gamma^0)^\dagger = \gamma^0, \quad (\gamma^i)^\dagger = -\gamma^i \quad \Rightarrow \quad (\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0 \quad (3.25)$$

and therefore also $\gamma^0 M_{\mu\nu}^\dagger \gamma^0 = M_{\mu\nu}$ and $\gamma^0 D(\Lambda)^\dagger \gamma^0 = D(\Lambda)^{-1}$. This is why we define the **conjugate spinor**

$$\bar{\psi} := \psi^\dagger \gamma^0 \quad \Rightarrow \quad \bar{\psi}'(x') = \psi^\dagger(x) D(\Lambda)^\dagger \gamma^0 = \psi^\dagger(x) \gamma^0 D(\Lambda)^{-1} = \bar{\psi}(x) D(\Lambda)^{-1}, \quad (3.26)$$

because it makes the quantity $\bar{\psi}\psi$ invariant:

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x) D(\Lambda)^{-1} D(\Lambda) \psi(x) = \bar{\psi}(x)\psi(x). \quad (3.27)$$

Similarly, one can use the identity $D(\Lambda)^{-1} \gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu$ to show that $\bar{\psi}\gamma^\mu\psi$ transforms like a Lorentz vector: (Ex)

$$\bar{\psi}'(x') \gamma^\mu \psi'(x') = \bar{\psi}(x) D(\Lambda)^{-1} \gamma^\mu D(\Lambda) \psi(x) = \Lambda^\mu{}_\nu \bar{\psi}(x) \gamma^\nu \psi(x). \quad (3.28)$$

Moreover, when we contract a Lorentz vector with another one, we get a Lorentz scalar:

$$\begin{aligned} \bar{\psi}'(x') \gamma^\mu \partial'_\mu \psi'(x') &= \bar{\psi}(x) D(\Lambda)^{-1} \gamma^\mu (\Lambda^{-1})^\nu{}_\mu \partial_\nu D(\Lambda) \psi(x) \\ &= (\Lambda^{-1})^\nu{}_\mu \Lambda^\mu{}_\rho \bar{\psi}(x) \gamma^\rho \partial_\nu \psi(x) \\ &= \bar{\psi}(x) \gamma^\nu \partial_\nu \psi(x). \end{aligned} \quad (3.29)$$

From now on we will use the Feynman slash notation $\not{A} = \gamma^\mu A_\mu$ for a generic four-vector A_μ , so the last expression simply becomes $\bar{\psi} \not{\partial} \psi$. The definition also entails $\not{A}^2 = A^2$. Note that only the combinations $\bar{\psi} \not{A} \psi$ are Lorentz-invariant but not \not{A} itself. (Also, be careful with derivatives because $\not{A} = \gamma^\mu A_\mu = \gamma^0 A^0 - \boldsymbol{\gamma} \cdot \mathbf{A}$ whereas $\not{\partial} = \gamma^\mu \partial_\mu = \gamma^0 \partial^0 + \boldsymbol{\gamma} \cdot \boldsymbol{\nabla}$.) Finally, one can show that the bilinears

$$\bar{\psi} i\gamma^5 \psi, \quad \bar{\psi} \boldsymbol{\gamma} \boldsymbol{\gamma}^5 \psi, \quad \bar{\psi} \boldsymbol{\sigma}^{\mu\nu} \psi \quad (3.30)$$

transform like a pseudoscalar, axialvector and tensor, respectively. We will discuss this later in the context of discrete symmetries.

Dirac Lagrangian. The simplest Lorentz scalars that we can build from $\psi(x)$ and $\bar{\psi}(x)$ and include non-trivial dynamics are $\bar{\psi}\psi$ and $\bar{\psi}\not{\partial}\psi$. Unlike in the scalar case, we can construct a Lorentz-invariant action already with those two terms alone (which contain only one derivative):

$$S = \int d^4x \mathcal{L} = \int d^4x \bar{\psi}(x) (i\not{\partial} - m) \psi(x). \quad (3.31)$$

The factor i is necessary to make \mathcal{L} real, the dimension of the field is $[\psi] = 3/2$, and m is a mass. Since $\psi(x)$ is a complex field we treat ψ and ψ^\dagger (or equivalently $\bar{\psi}$) as independent when deriving the Euler-Lagrange equations of motion:

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = (i\not{\partial} - m)\psi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0 \quad \Rightarrow \quad (i\not{\partial} - m)\psi = 0. \quad (3.32)$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = -m\bar{\psi}, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = i\bar{\psi}\overleftarrow{\gamma}^\mu \quad \Rightarrow \quad \bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0, \quad (3.33)$$

where $\overleftarrow{\partial}_\mu$ means that the derivative acts to the *left* instead of the right. When we take the resulting **Dirac equation** $(i\not{\partial} - m)\psi = 0$ and apply $(i\overleftarrow{\not{\partial}} + m)$ from the left, we obtain Klein-Gordon equations for each component of the Dirac field:

$$(i\overleftarrow{\not{\partial}} + m)(i\not{\partial} - m)\psi = -(\square + m^2)\psi = 0. \quad (3.34)$$

However, since the Dirac equation is a first-order equation it provides a *stronger* constraint on $\psi(x)$ than the KG equation, which is of second order.

Symmetries and currents. We can adapt the discussion of the Noether theorem to spinor fields without any modifications. The Noether current of Eq. (1.40) takes the form

$$-\delta j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi}_\alpha)} \delta \bar{\psi}_\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_\alpha)} \delta \psi_\alpha - T^{\mu\nu} \delta x_\nu = i\bar{\psi} \gamma^\mu \delta \psi - T^{\mu\nu} \delta x_\nu, \quad (3.35)$$

with the energy-momentum tensor of the Dirac field given by

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi}_\alpha)} \partial^\nu \bar{\psi}_\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_\alpha)} \partial^\nu \psi_\alpha - g^{\mu\nu} \mathcal{L} = i\bar{\psi} \gamma^\mu \partial^\nu \psi - g^{\mu\nu} \mathcal{L}. \quad (3.36)$$

- Let's start with **translation invariance**: each component of the Dirac field behaves like a scalar under translations, $\psi'_\alpha(x+a) = \psi_\alpha(x) \Leftrightarrow \delta \psi_\alpha = 0$ and $\delta x^\mu = a^\mu$, and therefore the conserved current is the energy-momentum tensor itself: $\partial_\mu T^{\mu\nu} = 0$. This can be easily checked: $\mathcal{L} = 0$ for solutions of the Dirac equation, so the last term in Eq. (3.36) vanishes, and the derivative of the first term also becomes zero when the Dirac equations are inserted. The conserved charges are the Hamiltonian of the Dirac field and its total momentum:

$$\begin{aligned} H &= \int d^3x T^{00} = \int d^3x \bar{\psi} (i\gamma^0 \partial^0 - i\not{\partial} + m) \psi = \int d^3x \bar{\psi} (-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m) \psi, \\ P^k &= \int d^3x T^{0k} = \int d^3x \psi^\dagger i\partial^k \psi = \int d^3x \psi^\dagger (-i\nabla_k) \psi. \end{aligned} \quad (3.37)$$

- The implications of **Lorentz invariance** can be worked out in a similar fashion. Lorentz transformations have the form

$$\begin{aligned} x' &= \Lambda x & \Leftrightarrow & \delta x^\mu = \varepsilon^{\mu\nu} x_\nu \\ \psi'(\Lambda x) &= D(\Lambda) \psi(x) & & \delta \psi = i\varepsilon_{\mu\nu} M^{\mu\nu} \psi \end{aligned} \quad (3.38)$$

and the infinitesimal current becomes

$$\begin{aligned} -\delta j^\mu &= -\frac{1}{2} \varepsilon_{\alpha\beta} \bar{\psi} \gamma^\mu M^{\alpha\beta} \psi - T^{\mu\alpha} \varepsilon_{\alpha\beta} x^\beta \\ &= -\frac{1}{2} \varepsilon_{\alpha\beta} \underbrace{(\bar{\psi} \gamma^\mu M^{\alpha\beta} \psi + T^{\mu\alpha} x^\beta - T^{\mu\beta} x^\alpha)}_{=: m^{\mu,\alpha\beta}}. \end{aligned} \quad (3.39)$$

The angular momentum density $m^{\mu,\alpha\beta}$ is the analogue of Eq. (1.44) from the scalar case and it is conserved: $\partial_\mu m^{\mu,\alpha\beta} = 0$. However, now it contains an additional spin contribution. Using the definition of $L^{\alpha\beta}$ in Eq. (1.44), we write

$$T^{\mu\alpha} x^\beta - T^{\mu\beta} x^\alpha = \bar{\psi} \gamma^\mu L^{\alpha\beta} \psi + (x^\alpha g^{\mu\beta} - x^\beta g^{\mu\alpha}) \mathcal{L}, \quad (3.40)$$

which allows us to combine the spin part $M^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$ with the orbital part $L^{\mu\nu}$ into a total angular momentum tensor $J^{\mu\nu} = L^{\mu\nu} + M^{\mu\nu}$:

$$m^{\mu,\alpha\beta} = \bar{\psi} \gamma^\mu J^{\alpha\beta} \psi + (x^\alpha g^{\mu\beta} - x^\beta g^{\mu\alpha}) \mathcal{L}. \quad (3.41)$$

Consider for example the invariance under rotations: the corresponding generator for $M^{\mu\nu}$ is $\mathbf{\Sigma}/2$, and its analogue for $L^{\mu\nu}$ is the three-vector $\mathbf{L} = \mathbf{x} \times (-i\nabla)$. Hence, the quantity that is conserved under rotations is the total angular momentum $\tilde{\mathbf{J}}$ of the field:

$$\int d^3x m^{0,ij} = \int d^3x \psi^\dagger J^{ij} \psi =: -\varepsilon_{ijk} \tilde{J}^k, \quad \tilde{\mathbf{J}} = \int d^3x \psi^\dagger \left(\mathbf{L} + \frac{\mathbf{\Sigma}}{2} \right) \psi. \quad (3.42)$$

- An example for **internal symmetries** is the $U(1)$ transformation

$$\psi' = e^{i\varepsilon} \psi, \quad \bar{\psi}' = e^{-i\varepsilon} \bar{\psi} \quad \Rightarrow \quad \delta \psi = i\varepsilon \psi, \quad \delta \bar{\psi} = -i\varepsilon \bar{\psi}, \quad (3.43)$$

with $\varepsilon \in \mathbb{R}$ constant, which leaves the Dirac Lagrangian invariant. It leads to the conserved **vector current** and charge

$$j_V^\mu = \bar{\psi} \gamma^\mu \psi, \quad Q_V = \int d^3x \psi^\dagger \psi. \quad (3.44)$$

- Another less obvious symmetry is the **axial** $U(1)_A$ symmetry

$$\psi' = e^{i\varepsilon \gamma_5} \psi, \quad \bar{\psi}' = \psi'^\dagger \gamma^0 = \psi^\dagger e^{-i\varepsilon \gamma_5} \gamma^0 = \psi^\dagger \gamma^0 e^{+i\varepsilon \gamma_5} = \bar{\psi} e^{i\varepsilon \gamma_5}, \quad (3.45)$$

which is only realized in the massless limit ($m = 0$) because the Lagrangian transforms as

$$\bar{\psi} (i\cancel{\partial} - m) \psi \quad \rightarrow \quad \bar{\psi} e^{i\varepsilon \gamma_5} (i\cancel{\partial} - m) e^{i\varepsilon \gamma_5} \psi = \bar{\psi} (i\cancel{\partial} - m e^{2i\varepsilon \gamma_5}) \psi. \quad (3.46)$$

In these rearrangements the relation $e^{i\varepsilon\gamma_5} = \cos\varepsilon + i\gamma_5\sin\varepsilon$ is helpful, which holds because $(\gamma_5)^2 = \mathbf{1}$, and we also used $\gamma_5 = \gamma_5^\dagger$ and $\gamma_5\gamma^\mu = -\gamma^\mu\gamma_5$. The corresponding axialvector current is

$$j_A^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi \quad (3.47)$$

and we can check explicitly that it is only conserved for $m = 0$:

$$\partial_\mu j_A^\mu = \bar{\psi}\overleftarrow{\not{\partial}}\gamma_5\psi - \bar{\psi}\gamma_5\not{\partial}\psi = 2im\bar{\psi}\gamma_5\psi. \quad (3.48)$$

This identity goes by the name **PCAC relation** (partially conserved axialvector current). Its underlying origin is that the left- and right-handed fields ψ_L , ψ_R decouple for $m = 0$, which leads to an enlarged *chiral symmetry* of the Lagrangian (see discussion below). Chiral symmetry has a rather prominent status in QCD: in a theory with N fermion flavors, the massless Lagrangian is invariant under $U(1)_V \times SU(N)_V \times SU(N)_A \times U(1)_A$. The latter two are explicitly broken by the quark masses, but $SU(N)_A$ is also *spontaneously* broken (which entails that the pions are Goldstone bosons), whereas $U(1)_A$ is *anomalously* broken at the quantum level.

Massless fields. Let's rewrite the Dirac Lagrangian (3.31) in terms of Weyl spinors. From Eq. (3.15) we have

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad \bar{\psi} = \psi^\dagger\gamma^0 = (\psi_R^\dagger, \psi_L^\dagger), \quad \not{\partial} = \begin{pmatrix} 0 & \sigma \cdot \partial \\ \bar{\sigma} \cdot \partial & 0 \end{pmatrix} \quad (3.49)$$

and the Dirac Lagrangian becomes

$$\mathcal{L} = i\psi_R^\dagger\sigma \cdot \partial\psi_R + i\psi_L^\dagger\bar{\sigma} \cdot \partial\psi_L - m(\psi_R^\dagger\psi_L + \psi_L^\dagger\psi_R). \quad (3.50)$$

If $m = 0$, the left- and right-handed spinors decouple and describe independent degrees of freedom, which is why the limit $m = 0$ is also called **chiral limit**. The corresponding Euler-Lagrange equations are the **Weyl equations**: $i\sigma \cdot \partial\psi_R = 0$, $i\bar{\sigma} \cdot \partial\psi_L = 0$. With the ansatz $\psi_{R,L}(x) = u_{R,L}(p)e^{-ipx}$ they become

$$\begin{aligned} (p \cdot \sigma) u_R = 0 \\ (p \cdot \bar{\sigma}) u_L = 0 \end{aligned} \quad \Rightarrow \quad h u_{R,L} = \pm \frac{1}{2} u_{R,L} \quad \text{with} \quad h = \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{|\mathbf{p}|}, \quad (3.51)$$

where h is the **helicity** (the projection of the spin in the momentum direction). Hence, in the limit $m = 0$ the right- and left-handed Weyl spinors are eigenstates of the helicity with eigenvalues $\pm\frac{1}{2}$. If $m \neq 0$, they no longer decouple and it is impossible to define a Lorentz-invariant notion of helicity: in that case particles travel with velocity $v < c$ and it is always possible to find a Lorentz frame where the particle moves in the opposite direction, which causes a change in the helicity.

In the chiral limit $m = 0$ the helicity is Lorentz-invariant (and actually even Poincaré-invariant). That is, in principle we could interpret the two helicity states

ψ_L and ψ_R (whose dynamics also decouple) as two different species of particles. However, parity still transforms them into each other and hence we need both to have a parity-invariant Lagrangian. For this reason we do not identify them as independent degrees of freedom but rather as two polarization states of the same particle. The exception are theories that *break* parity invariance, because in that case it is not necessary to have both chiralities. For example, the right-handed neutrinos do not participate in the weak interaction, and neutrinos in the Standard Model are therefore described by massless left-handed Weyl fields with Lagrangian $i\psi_L^\dagger \bar{\sigma} \cdot \partial \psi_L$.

A special case are **Majorana spinors** where ψ_L and ψ_R are not independent but $\psi_R = i\sigma^2 \psi_L^*$. The corresponding four-spinor $\psi = (\psi_L, \psi_R)$ is invariant under charge conjugation. This is the spinor analogue of the real scalar field (the condition $\psi = \psi^*$ alone is not Lorentz-invariant because $D(\Lambda)$ is not real), so the corresponding particle would be its own antiparticle. Since in that case we lose the $U(1)$ symmetry, Majorana fields cannot describe fermions that carry a $U(1)$ charge (electric charge, lepton number, etc.). Possible candidates are, again, the neutrinos whose masses are very small but most likely nonzero. If they were Majorana particles, lepton number symmetry would be violated, and experiments on neutrino-less double beta decay aim at detecting such violations.

In general it is quite useful to study massless Dirac particles because scattering matrices often simplify greatly if the particles can be approximated as massless. This is usually realized in QED processes because the electron mass is much smaller compared to other relevant scales. It is also useful in QCD where the light up and down quarks can be treated as nearly massless particles. An interesting feature in the chiral limit is that both fields ψ_L , ψ_R transform now independently under $U(1)$ transformations, which leave the Lagrangian separately invariant:

$$\psi'_L = e^{i\varepsilon_L} \psi_L, \quad \psi'_R = e^{i\varepsilon_R} \psi_R. \quad (3.52)$$

The corresponding $U(1)_L \times U(1)_R$ symmetry is called **chiral symmetry**. It is equivalent to the $U(1)_V \times U(1)_A$ symmetry that we encountered above because the conserved left- and right-handed currents are linear combinations of the vector and axialvector currents j_V^μ and j_A^μ .

(Ex)

Classical solutions of the Dirac equation. Like in the scalar case, the general solutions of the free Dirac equations can be expressed by plane waves with positive- and negative frequency modes:

$$\begin{aligned} \psi_+(x) &= u(\mathbf{p}) e^{-ipx} & \Rightarrow & \quad (\not{p} - m) u(\mathbf{p}) = 0, \\ \psi_-(x) &= v(\mathbf{p}) e^{ipx} & & \quad (\not{p} + m) v(\mathbf{p}) = 0. \end{aligned} \quad (3.53)$$

We recover $p^2 = m^2$ by multiplying the equations with $\not{p} \pm m$, so these are indeed solutions of the Dirac equation. We have again chosen $p^0 = +E_p = +\sqrt{\mathbf{p}^2 + m^2}$ to be positive and put the sign instead in the exponential; we could have also started with e^{-ipx} alone and distinguish the two solutions by $p^0 = \pm E_p$ (with a change $\mathbf{p} \rightarrow -\mathbf{p}$). The Dirac equation can be written in the form

$$\begin{aligned} (p^0 \gamma^0 - \mathbf{p} \cdot \boldsymbol{\gamma} - m) u(\mathbf{p}) &= 0 & \Rightarrow & \quad \gamma^0 (\mathbf{p} \cdot \boldsymbol{\gamma} + m) u(\mathbf{p}) = E_p u(\mathbf{p}), \\ (p^0 \gamma^0 - \mathbf{p} \cdot \boldsymbol{\gamma} + m) v(\mathbf{p}) &= 0 & & \quad \gamma^0 (\mathbf{p} \cdot \boldsymbol{\gamma} + m) v(-\mathbf{p}) = -E_p v(-\mathbf{p}). \end{aligned} \quad (3.54)$$

so that $u(\mathbf{p})$ and $v(-\mathbf{p})$ are eigenstates of the Dirac Hamiltonian with eigenvalues $\pm E_p$.

In the chiral representation we can write $u = (u_L, u_R)$, and with the explicit form of the γ -matrices in Eq. (3.15) the Dirac equation becomes

$$\begin{pmatrix} -m & p \cdot \sigma \\ p \cdot \bar{\sigma} & -m \end{pmatrix} \begin{pmatrix} u_L \\ u_R \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{aligned} (p \cdot \sigma) u_R &= m u_L, \\ (p \cdot \bar{\sigma}) u_L &= m u_R. \end{aligned} \quad (3.55)$$

These two equations are consistent because $(p \cdot \sigma)(p \cdot \bar{\sigma}) = p^2 = m^2$:

$$(p \cdot \sigma)(p \cdot \bar{\sigma}) = p_0^2 - p^i p^j \sigma^i \sigma^j = p_0^2 - \mathbf{p}^2 = p^2 = m^2. \quad (3.56)$$

(Use $\{\sigma^i, \sigma^j\} = 2\delta_{ij}$). Note that ψ_L and ψ_R are no longer helicity eigenstates because of the mass term. Instead, their solution can be written as

$$u_L = \sqrt{p \cdot \bar{\sigma}} \xi, \quad u_R = \sqrt{p \cdot \sigma} \xi, \quad (3.57)$$

where ξ_s with $s = \pm 1$ are two-component spinors that we normalize to $\xi_s^\dagger \xi_{s'} = \delta_{ss'}$. The analogous analysis for negative-frequency modes gives

$$v_L = \sqrt{p \cdot \sigma} \eta, \quad v_R = -\sqrt{p \cdot \bar{\sigma}} \eta, \quad (3.58)$$

so that we obtain in total

$$u_s(\mathbf{p}) = \begin{pmatrix} \sqrt{p \cdot \bar{\sigma}} \xi_s \\ \sqrt{p \cdot \sigma} \xi_s \end{pmatrix}, \quad v_s(\mathbf{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma} \eta_s \\ -\sqrt{p \cdot \bar{\sigma}} \eta_s \end{pmatrix}. \quad (3.59)$$

The two components of s can be interpreted as the spin direction. For example, if we choose the basis for the two-component spinors ξ_s as

$$\xi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (3.60)$$

they are eigenvectors of the spin matrix $\sigma^3/2$ with eigenvalues $\pm \frac{1}{2}$, so they describe spinors with spin $\pm \frac{1}{2}$ in z -direction.

Using the explicit form of the spinors, it is easy to prove the orthogonality relations

(Ex)
$$\begin{aligned} \bar{u}_s(\mathbf{p}) u_{s'}(\mathbf{p}) &= 2m \delta_{ss'}, & \bar{u}_s(\mathbf{p}) v_{s'}(\mathbf{p}) &= 0, & u_s^\dagger(\mathbf{p}) u_{s'}(\mathbf{p}) &= 2E_p \delta_{ss'}, \\ \bar{v}_s(\mathbf{p}) v_{s'}(\mathbf{p}) &= -2m \delta_{ss'}, & \bar{v}_s(\mathbf{p}) u_{s'}(\mathbf{p}) &= 0, & v_s^\dagger(\mathbf{p}) v_{s'}(\mathbf{p}) &= 2E_p \delta_{ss'}, \end{aligned} \quad (3.61)$$

as well as the completeness relations

$$\sum_s u_s(\mathbf{p}) \bar{u}_s(\mathbf{p}) = \not{p} + m, \quad \sum_s v_s(\mathbf{p}) \bar{v}_s(\mathbf{p}) = \not{p} - m. \quad (3.62)$$

Be careful because $u_s^\dagger(\mathbf{p}) v_{s'}(\mathbf{p}) \neq 0$ and $v_s^\dagger(\mathbf{p}) u_{s'}(\mathbf{p}) \neq 0$, but instead one has

$$u_s^\dagger(\mathbf{p}) v_{s'}(-\mathbf{p}) = v_s^\dagger(\mathbf{p}) u_{s'}(-\mathbf{p}) = 0. \quad (3.63)$$

The general solutions of the Dirac equation can be written as

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} \sum_s (a_s(\mathbf{p}) u_s(\mathbf{p}) e^{-ipx} + b_s^*(\mathbf{p}) v_s(\mathbf{p}) e^{ipx})_{p^0=E_p}. \quad (3.64)$$

(Ex)

If we define the positive- and negative-energy projectors

$$\Lambda_{\pm}(\mathbf{p}) = \frac{1}{2} \left(\mathbb{1} \pm \frac{\not{\mathbf{p}}}{m} \right) = \frac{\pm \not{\mathbf{p}} + m}{2m}, \quad \Rightarrow \quad \begin{aligned} \Lambda_{\pm}(\mathbf{p})^2 &= \Lambda_{\pm}(\mathbf{p}), \\ \Lambda_{\pm}(\mathbf{p}) \Lambda_{\mp}(\mathbf{p}) &= 0 \end{aligned} \quad (3.65)$$

and write $w_s^+(\mathbf{p}) = u_s(\mathbf{p})$ and $w_s^-(\mathbf{p}) = v_s(\mathbf{p})$, then the Dirac equation simply becomes

$$\Lambda_{\mp}(\mathbf{p}) w_s^{\pm}(\mathbf{p}) = 0, \quad \Lambda_{\pm}(\mathbf{p}) w_s^{\pm}(\mathbf{p}) = w_s^{\pm}(\mathbf{p}) \quad (3.66)$$

and Eqs. (3.61–3.62) take the compact form

$$\bar{w}_s^{\pm}(\mathbf{p}) w_{s'}^{\pm}(\mathbf{p}) = \pm 2m \delta_{ss'}, \quad \sum_s w_s^{\pm}(\mathbf{p}) \bar{w}_s^{\pm}(\mathbf{p}) = 2m \Lambda_{\pm}(\mathbf{p}). \quad (3.67)$$

We can derive more useful relations by adding and subtracting the Dirac equations for w^{\pm} and \bar{w}^{\pm} :

$$\begin{aligned} (\not{\mathbf{p}} \mp m) w^{\pm} = 0 &\Rightarrow \bar{w}^{\pm} \mathcal{O} (\not{\mathbf{p}} \mp m) w^{\pm} = 0 &\Rightarrow \bar{w}^{\pm} \{ \mathcal{O}, \not{\mathbf{p}} \} w^{\pm} = \pm 2m \bar{w}^{\pm} \mathcal{O} w^{\pm}, \\ \bar{w}^{\pm} (\not{\mathbf{p}} \mp m) = 0 &\Rightarrow \bar{w}^{\pm} (\not{\mathbf{p}} \mp m) \mathcal{O} w^{\pm} = 0 &\Rightarrow \bar{w}^{\pm} [\mathcal{O}, \not{\mathbf{p}}] w^{\pm} = 0, \end{aligned} \quad (3.68)$$

where \mathcal{O} is some combination of Dirac matrices. For example, it follows that

$$\bar{w}^{\pm} \gamma_5 w^{\pm} = 0, \quad \bar{w}^{\pm} \gamma^{\mu} w^{\pm} = 2p^{\mu}, \quad \text{etc.} \quad (3.69)$$

In the **standard representation** we write $u = (\phi, \chi)$, and with the explicit form of the γ -matrices in Eq. (3.19) the Dirac equation for $u(\mathbf{p})$ becomes

$$\begin{pmatrix} E_p - m & -\mathbf{p} \cdot \boldsymbol{\sigma} \\ \mathbf{p} \cdot \boldsymbol{\sigma} & -(E_p + m) \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{aligned} (\mathbf{p} \cdot \boldsymbol{\sigma}) \chi &= (E_p - m) \phi, \\ (\mathbf{p} \cdot \boldsymbol{\sigma}) \phi &= (E_p + m) \chi. \end{aligned} \quad (3.70)$$

This is again consistent because $(\mathbf{p} \cdot \boldsymbol{\sigma})(\mathbf{p} \cdot \boldsymbol{\sigma}) = \mathbf{p}^2 = E_p^2 - m^2 = (E_p + m)(E_p - m)$, and the solution can be written as

$$u_s(\mathbf{p}) = \sqrt{E_p + m} \begin{pmatrix} \xi_s \\ \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{E_p + m} \xi_s \end{pmatrix}, \quad v_s(\mathbf{p}) = \sqrt{E_p + m} \begin{pmatrix} \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{E_p + m} \eta_s \\ \eta_s \end{pmatrix}. \quad (3.71)$$

The standard representation is convenient because in the rest frame ($\mathbf{p} = 0$, $E_p = m$) only the upper component of $u_s(\mathbf{p})$ and the lower component of $v_s(\mathbf{p})$ survives, which correspond to the positive- and negative-energy eigenstates. Therefore it is also useful for describing a nonrelativistic particle with $v \ll c$ where the lower component of $u_s(\mathbf{p})$ can be neglected. This is the essential difference between the chiral representation, where the upper and lower components separate left- and right-handedness, and the standard representation where they are related to positive and negative energies.

Classical field theory vs. quantum mechanics. In the spirit of the scalar field we could equip the solutions of the Dirac equation with a scalar product,

$$\langle \psi_1, \psi_2 \rangle := \int d\sigma_{\mu} \bar{\psi}_1(x) \gamma^{\mu} \psi_2(x) = \int d^3x \psi_1^{\dagger}(x) \psi_2(x), \quad (3.72)$$

whose norm $\langle \psi | \psi \rangle$ is again the $U(1)$ charge. By doing so we entered relativistic quantum mechanics: if we interpret the field $\psi(x)$ as the wave function of a single particle, whose scalar product is Eq. (3.72), then the quantities H , \mathbf{P} and $\hat{\mathbf{J}}$ in Eqs. (3.37) and (5.21) can be interpreted as the expectation values of the Hamilton operator $\boldsymbol{\gamma} \cdot (-i\nabla) + m$, the momentum operator $-i\nabla$, and the angular momentum operator $\mathbf{x} \times (-i\nabla) + \frac{\boldsymbol{\Sigma}}{2}$,

respectively. In that sense, quantum mechanics *is* essentially classical field theory, except that the additional scalar product also allows for a probability interpretation of the field. In contrast to the scalar field, the scalar product is indeed positive definite because when we insert the Dirac solutions the $U(1)$ charge takes the form

$$\langle \psi, \psi \rangle = \int d^3x \psi^\dagger(x) \psi(x) = \int \frac{d^3p}{2E_p} \sum_s (|a_{p,s}|^2 + |b_{p,s}|^2). \quad (3.73)$$

In exchange, the Hamiltonian is no longer positive definite and permits negative-energy eigenvalues.

In quantum field theory we omit the single-particle interpretation but rather view $\psi(x)$, $\bar{\psi}(x)$ as field operators on the Fock space. The quantities H , \mathbf{P} and $\tilde{\mathbf{J}}$ then become the Hamilton, momentum and angular-momentum operators of the field, and their eigenvalues are the total energy, momentum and angular momentum of some multiparticle state. After quantization with anticommutators, the situation above is also reversed: the Hamiltonian becomes positive but the $U(1)$ charge is no longer positive definite. This is no reason to worry because the charge is no longer interpreted as a probability; it is the number operator that counts the number of particles *minus* antiparticles in a state.

There is another piece of insight that we can take away from the discussion: since the structure of quantum mechanics is basically that of classical field theory, it reflects the ‘classical’ *tree-level* contributions to quantum processes, whereas loop corrections are reserved for the quantum field-theoretical treatment. In QED the electromagnetic coupling is so small that tree-level diagrams already provide a good approximation — which explains the successes of quantum mechanics in describing electrons, photons, and the physics of atoms and molecules.

4 Quantization of the Dirac field

The quantization of the Dirac field proceeds almost along the same lines as that for the classical field, except for one important difference: instead of commutation relations for the fields we will need *anticommutation* relations to ensure a positive spectrum.

Quantized Hamiltonian. To see this, let's calculate the Hamiltonian without imposing any commutation relations yet. We start with the general solutions $\psi(x)$, $\bar{\psi}(x)$ of the Dirac equation, which we reinterpret as operators on a state space:

$$\begin{aligned}\psi(x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} \sum_s \left(a_{p,s} u_{p,s} e^{-ipx} + b_{p,s}^\dagger v_{p,s} e^{ipx} \right)_{p^0=E_p}, \\ \bar{\psi}(x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} \sum_s \left(b_{p,s} \bar{v}_{p,s} e^{-ipx} + a_{p,s}^\dagger \bar{u}_{p,s} e^{ipx} \right)_{p^0=E_p}.\end{aligned}\quad (4.1)$$

The coefficients $a_{p,s} = a_s(\mathbf{p})$ and $b_{p,s} = b_s(\mathbf{p})$ inherit the operator structure, whereas $u_{p,s} = u_s(\mathbf{p})$ and $v_{p,s} = v_s(\mathbf{p})$ are the Dirac spinors that we worked out above. The conjugate momentum is

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \bar{\psi} i \gamma^0 = i \psi^\dagger, \quad (4.2)$$

and therefore the Hamiltonian becomes

$$H = \int d^3x \psi^\dagger(x) \gamma^0 (-i \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m) \psi(x). \quad (4.3)$$

This agrees with our earlier result (3.37) extracted from the energy momentum tensor. Note also that $\gamma^0(-i \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m)$ is the Dirac Hamiltonian that is well known from quantum mechanics.

When we insert the Fourier decomposition for the fields, then after some calculation (which is analogous to Eqs. (2.6–2.11)) we arrive at

$$H = \int \frac{d^3p}{2E_p} E_p \sum_s \left(a_{p,s}^\dagger a_{p,s} - b_{p,s} b_{p,s}^\dagger \right). \quad (4.4)$$

The calculation goes along the same lines as before: take the three-dimensional Fourier transform

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int d^3p \tilde{\psi}_p(t) e^{i\mathbf{p} \cdot \mathbf{x}}, \quad \tilde{\psi}_p(t) = \frac{1}{2E_p} \left(a_{p,s} u_{p,s} e^{-iE_p t} + b_{-p,s}^\dagger v_{-p,s} e^{iE_p t} \right) \quad (4.5)$$

and plug it into the Hamiltonian, which in momentum space becomes

$$H = \int d^3p \tilde{\psi}_p^\dagger(t) \gamma^0 (\mathbf{p} \cdot \boldsymbol{\gamma} + m) \tilde{\psi}_p(t). \quad (4.6)$$

From Eq. (3.54) we know that the solutions of the Dirac equation are eigenfunctions of the Dirac Hamiltonian, which simplifies the calculations a lot:

$$\gamma^0 (\mathbf{p} \cdot \boldsymbol{\gamma} + m) \tilde{\psi}_p(t) = \frac{1}{2} \left(a_{p,s} u_{p,s} e^{-iE_p t} - b_{-p,s}^\dagger v_{-p,s} e^{iE_p t} \right). \quad (4.7)$$

With the orthogonality relations (3.61) and (3.63), the time dependencies cancel and one arrives at the result above.

(Ex)

Actually, the result in Eq. (4.4) looks rather suspicious because of the minus sign. Suppose we postulate canonical commutation relations:

$$[\psi_\alpha(x), \psi_\beta^\dagger(y)]_{x^0=y^0} \stackrel{!}{=} \delta_{\alpha\beta} \delta^3(\mathbf{x} - \mathbf{y}). \quad (4.8)$$

Then the corresponding commutator relations in momentum space would read

$$[a_{p,s}, a_{p',s'}^\dagger] = [b_{p,s}^\dagger, b_{p',s'}] = 2E_p \delta_{ss'} \delta^3(\mathbf{p} - \mathbf{p}'), \quad (4.9)$$

with all other commutators zero, which is easy to verify by inserting the Fourier decomposition into Eq. (4.8). In that way, once we subtract the vacuum energy, the Hamiltonian is proportional to $a^\dagger a - b^\dagger b$ and therefore the energy

$$\langle \lambda | H | \lambda \rangle = \int \frac{d^3 p}{2E_p} E_p \langle \lambda | a_{p,s}^\dagger a_{p,s} - b_{p,s}^\dagger b_{p,s} | \lambda \rangle \quad (4.10)$$

is unbounded from below. We could ensure that the energy is positive by demanding a negative norm, $\|b_{p,s} | \lambda \rangle\|^2 < 0$, but this violates unitarity. So apparently we face a dilemma: either we have an unstable vacuum (negative energies) or we violate unitarity of the theory (negative norms).

The correct way to resolve the problem is to impose **anticommutation relations**:

$$\{\psi_\alpha(x), \psi_\beta^\dagger(y)\}_{x^0=y^0} = \delta_{\alpha\beta} \delta^3(\mathbf{x} - \mathbf{y}), \quad (4.11)$$

with all other *anticommutators* zero, which entails

$$\{a_{p,s}, a_{p',s'}^\dagger\} = \{b_{p,s}^\dagger, b_{p',s'}\} = 2E_p \delta_{ss'} \delta^3(\mathbf{p} - \mathbf{p}'), \quad (4.12)$$

again with all other anticommutators zero. In that case the second term in (4.4) picks up a minus sign, and after throwing away the infinite constant the normal-ordered Hamiltonian is again positive:

$$H = \int \frac{d^3 p}{2E_p} E_p \sum_s \left(a_{p,s}^\dagger a_{p,s} + b_{p,s}^\dagger b_{p,s} \right). \quad (4.13)$$

In that way the normal ordering for fermions introduces a minus sign for each interchange of operators. The same result follows for the four-momentum operator:

$$P^\mu = \int \frac{d^3 p}{2E_p} p^\mu \sum_s \left(a_{p,s}^\dagger a_{p,s} + b_{p,s}^\dagger b_{p,s} \right). \quad (4.14)$$

Fock space and Fermi-Dirac statistics. Despite the anticommutation relation for the fields, the commutation relations (2.24) for the momentum operator still hold as a consequence of the identity $[AB, C] = A\{B, C\} - \{A, C\}B$. Hence we can take over the analysis from the scalar field: the vacuum is still defined by $a_{p,s} | 0 \rangle = b_{p,s} | 0 \rangle = 0$, multi-particle states are obtained by acting on the vacuum with $a_{p,s}^\dagger$ or $b_{p,s}^\dagger$, and their normalization is the same as before. Note in particular that the norm is positive:⁴

$$\langle 0 | a_{p,s} a_{p',s'}^\dagger | 0 \rangle = \langle 0 | b_{p,s} b_{p',s'}^\dagger | 0 \rangle = 2E_p \delta_{ss'} \delta^3(\mathbf{p} - \mathbf{p}'). \quad (4.15)$$

⁴Remember that $\delta^3(\mathbf{0})$ is proportional to the volume, so this infrared divergence is not a serious problem. Had we worked with smeared operators from the beginning (at the expense of a simple notation), the norm would be well-defined.

As before, the eigenvalue of the momentum operator P^μ is the total momentum of the state.

However, there is one important difference: since these operators anticommute between themselves, an N -particle state is *antisymmetric* under particle exchange:

$$a_{p,s}^\dagger a_{q,r}^\dagger |0\rangle = -a_{q,r}^\dagger a_{p,s}^\dagger |0\rangle. \quad (4.16)$$

Therefore, spin- $\frac{1}{2}$ particles are fermions, i.e., they obey Fermi-Dirac statistics. In particular, they satisfy the Pauli principle: no two fermionic states of exactly the same quantum numbers are possible, because we can never create more than one particle in the same state:

$$\{a_{p,s}^\dagger, a_{p,s}^\dagger\} = 0 \quad \Rightarrow \quad a_{p,s}^\dagger a_{p,s}^\dagger |0\rangle = 0. \quad (4.17)$$

This is another manifestation of the **spin-statistics theorem**: Lorentz invariance, positive energies, unitarity (=positive norms) and causality together imply that particles with integer spin obey Bose-Einstein statistics, whereas particles with half-odd integer spin obey Fermi-Dirac statistics. By working out the $U(1)$ charge from Eq. (3.44),

$$Q = \int d^3x :\psi^\dagger \psi: = \int \frac{d^3p}{2E_p} \sum_s \left(a_{p,s}^\dagger a_{p,s} - b_{p,s}^\dagger b_{p,s} \right), \quad (4.18)$$

we arrive at the same interpretation as for the complex scalar field: $a_{p,s}^\dagger$ and $b_{p,s}^\dagger$ create fermions and antifermions, respectively, and the charge equals the number of particles minus antiparticles. Note that the minus sign in Q is also a consequence of the anticommutation relations: Q was non-negative in the classical theory, where it could be interpreted as a scalar product between fields, cf. Eq. (3.73).

The spin operator that follows from the classical Noether charge (5.21) is given by

$$\int d^3x :\psi^\dagger \frac{\Sigma}{2} \psi: . \quad (4.19)$$

One can show (Peskin-Schroeder, p.61) that applying it to a state $a_{p,s}^\dagger |0\rangle$ gives eigenvalue $s/2$ whereas applied to $b_{p,s}^\dagger |0\rangle$ it gives eigenvalue $-s/2$, where $s = \pm 1$. Therefore, $a_{p,s}^\dagger |0\rangle$ describes a fermion (for example an electron) with mass m , energy E_p , spin $\frac{1}{2}$ and spin polarization $s/2$, whereas $b_{p,s}^\dagger |0\rangle$ describes an antifermion (positron) with mass m , energy E_p , spin $\frac{1}{2}$ and spin polarization $-s/2$. The state $\bar{\psi}(x) |0\rangle$ describes a fermion at position x and $\psi(x) |0\rangle$ an antifermion at position x .

Causality. Despite the anticommutator relations that we imposed for the Dirac fields, the microcausality axiom must remain unchanged: all physical observables are bosonic operators and must commute at spacelike distances,

$$[\mathcal{O}_1(x), \mathcal{O}_2(y)] \stackrel{!}{=} 0 \quad \text{if } (x-y)^2 < 0. \quad (4.20)$$

This is ensured by requiring

$$S_{\alpha\beta}(x-y) := \{\psi_\alpha(x), \bar{\psi}_\beta(y)\} \stackrel{!}{=} 0 \quad \text{if } (x-y)^2 < 0, \quad (4.21)$$

which is the generalization of Eq. (2.74) in the scalar case. Eq. (4.20) can be checked directly for fermion bilinears $\mathcal{O}_i(x) = \bar{\psi}(x) \Gamma_i \psi(x)$, where Γ_i is any of the Dirac matrices in Eq. (3.21), by exploiting the identity

(Ex)

$$[AB, CD] = A \{B, C\} D - C \{A, D\} B - \frac{\{A, C\} [B, D] + [A, C] \{B, D\}}{2}. \quad (4.22)$$

Inserting the Fourier decomposition (4.1), the anticommutator relation (4.12) and the completeness relations (3.62), this expression becomes

$$\begin{aligned} S(z) &= \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_p} ((\not{p} + m) e^{-ipz} + (\not{p} - m) e^{ipz}) \\ &= (i\not{\partial} + m) \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_p} (e^{-ipz} - e^{ipz}) = (i\not{\partial} + m) \Delta(z) \end{aligned} \quad (4.23)$$

where $\Delta(z)$ is the scalar analogue in Eq. (2.69). From here it is easy to recover our original commutator relations (4.11):

$$S(z)|_{z^0=0} = (i\not{\partial} + m) \Delta(z)|_{z^0=0} = \gamma^0 \delta^3(\mathbf{z}), \quad (4.24)$$

because $\partial_0 \Delta(z)|_{z^0=0} = -i\delta^3(\mathbf{z})$, $\partial_i \Delta(z)|_{z^0=0} = 0$ and $\Delta(z)|_{z^0=0} = 0$.

Feynman propagator. Similarly, we define the Feynman propagator for fermions as

$$S_F(x - y) := \langle 0 | \mathbb{T} \psi(x) \bar{\psi}(y) | 0 \rangle = \begin{cases} \langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle & \text{if } x^0 \geq y^0, \\ -\langle 0 | \bar{\psi}(y) \psi(x) | 0 \rangle & \text{if } y^0 \geq x^0, \end{cases} \quad (4.25)$$

with the crucial difference of the minus sign. It is necessary because if $(x - y)^2 < 0$ we have $S(x - y) = 0$ and therefore $\psi(x) \bar{\psi}(y) = -\bar{\psi}(y) \psi(x)$. For spacelike distances the question of whether $x^0 > y^0$ or $x^0 < y^0$ depends on the frame, and to arrive at a frame-independent definition of the time-ordering symbol \mathbb{T} the expression for $\mathbb{T} \psi(x) \bar{\psi}(y)$ for $x^0 > y^0$ and $x^0 < y^0$ must agree.

Using the definition above and inserting the Fourier decomposition, one evaluates $S_F(z) = (i\not{\partial} + m) \Delta_F(z)$ and therefore the fermion propagator becomes

$$S_F(z) = \int \frac{d^4p}{(2\pi)^4} e^{-ipz} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}. \quad (4.26)$$

Since $(\not{p} + m)(\not{p} - m) = p^2 - m^2$, the inverse propagator in momentum space has the form

$$S_F^{-1}(p) = -i(\not{p} - m). \quad (4.27)$$

The Feynman propagator is a Green function of the Dirac equation, i.e., it is one of the four possible solutions to the equation $(i\not{\partial} - m) G(z) = i\delta^4(z)$. Their interpretation and closure procedure in the complex plane are as in the scalar theory.

Parity. Earlier we have seen that the parity operation $x \rightarrow x' = (t, -\mathbf{x})$ exchanges the left- and right-handed Weyl spinors:

$$\begin{pmatrix} \psi'_L(x') \\ \psi'_R(x') \end{pmatrix} = \begin{pmatrix} \psi_R(x) \\ \psi_L(x) \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} \psi'(x') &= \gamma^0 \psi(x) \\ \bar{\psi}'(x') &= \bar{\psi}(x) \gamma^0. \end{aligned} \quad (4.28)$$

Consequently, the bilinears $\bar{\psi}\psi$ and $\bar{\psi}i\gamma_5\psi$ transform as scalars and pseudoscalars under parity (γ^0 anticommutes with γ_5):

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)\psi(x), \quad \bar{\psi}'(x')i\gamma_5\psi'(x') = -\bar{\psi}(x)i\gamma_5\psi(x). \quad (4.29)$$

The factor i is necessary to make the pseudoscalar bilinear real: $(\bar{\psi}i\gamma_5\psi)^\dagger = \bar{\psi}i\gamma_5\psi$. Likewise, $\bar{\psi}\gamma^\mu\psi$ and $\bar{\psi}\gamma^\mu\gamma_5\psi$ transform as vectors and axialvectors, respectively:

$$\bar{\psi}'(x')\gamma^\mu\psi'(x') = \pm\bar{\psi}(x)\gamma^\mu\psi(x) \quad \bar{\psi}'(x')\gamma^\mu\gamma_5\psi'(x') = \mp\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x), \quad (4.30)$$

where the upper sign corresponds to $\mu = 0$ and the lower one to $\mu = 1, 2, 3$.

How does parity act on the Fock space? If we introduce the unitary operator U_P that transforms a state as $|\lambda'\rangle = U_P|\lambda\rangle$, then the quantum version of Eq. (4.28) follows from the same reasoning as in Eq. (2.59):

$$U_P\psi(x)U_P^{-1} = \gamma^0\psi(x'), \quad U_P\bar{\psi}(x)U_P^{-1} = \bar{\psi}(x')\gamma^0. \quad (4.31)$$

We ignore possible phase factors for simplicity because they are not important for the discussion. Applied to the Fourier decomposition (4.1), we can work out the action of U_P on the creation and annihilation operators:

$$U_P a_{p,s} U_P^{-1} = a_{-p,s}, \quad U_P b_{p,s}^\dagger U_P^{-1} = -b_{-p,s}^\dagger. \quad (4.32)$$

To derive this, start with

$$\gamma^0\psi(x') = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} \sum_s \left(a_{p,s} (\gamma^0 u_{p,s}) e^{-iE_p t - \mathbf{p}\cdot\mathbf{x}} + b_{p,s}^\dagger (\gamma^0 v_{p,s}) e^{iE_p t + \mathbf{p}\cdot\mathbf{x}} \right). \quad (\text{Ex})$$

From Eq. (3.59) it follows that $\gamma^0 u_{p,s} = u_{-p,s}$ and $\gamma^0 v_{p,s} = -v_{-p,s}$; remember our shorthand notation $u_{p,s} = u_s(\mathbf{p})$, so the minus sign switches only the spatial components. Exchanging $\mathbf{p} \rightarrow -\mathbf{p}$ in the integral leads to

$$\gamma^0\psi(x') = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} \sum_s \left(a_{-p,s} u_{p,s} e^{-ipx} - b_{-p,s}^\dagger v_{p,s} e^{ipx} \right)_{p^0=E_p}, \quad (4.33)$$

and comparison with the direct expression for $U_P\psi(x)U_P^{-1}$ gives the result in Eq. (4.32).

Applied to one-particle and -antiparticle states, this entails

$$\begin{aligned} U_P |\mathbf{p}, s, a\rangle &= U_P a_{p,s}^\dagger |0\rangle = a_{-p,s}^\dagger |0\rangle = |-\mathbf{p}, s, a\rangle, \\ U_P |\mathbf{p}, s, b\rangle &= U_P b_{p,s}^\dagger |0\rangle = -b_{-p,s}^\dagger |0\rangle = -|-\mathbf{p}, s, b\rangle, \end{aligned} \quad (4.34)$$

where we assumed parity invariance of the vacuum $U_P|0\rangle = |0\rangle$. The relative minus sign tells us that fermions and antifermions carry opposite **intrinsic parity**. For scalar fields we would not get the relative minus sign: the intrinsic parity of a spin-0 particle and its antiparticle are equal.

Charge conjugation. As we remarked in the context of Majorana spinors, one cannot construct a charge-conjugate Dirac spinor in the form $\psi \rightarrow \psi^*$ because this is not Lorentz-invariant: since $D^*(\Lambda) \neq D(\Lambda)$, a Lorentz transformation will mix ψ and ψ^* . Instead, the property $\gamma^{\mu*} = \gamma^2 \gamma^\mu \gamma^2$ implies $D^*(\Lambda) = -\gamma^2 D(\Lambda) \gamma^2$, which allows us to define the operation of **charge conjugation** as

$$\psi^c = -i\gamma^2 \psi^*, \quad \bar{\psi}^c = i\bar{\psi}^* \gamma^2. \quad (4.35)$$

This is now indeed compatible with a Lorentz transformation:

$$(\psi^c)'(x') = -i\gamma^2 (\psi'(x'))^* = -i\gamma^2 D^*(\Lambda) \psi^*(x) = D(\Lambda) \psi^c(x). \quad (4.36)$$

Let's work this out in the chiral representation:

(Ex)

$$\gamma^2 D(\Lambda)^* \gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} D_L^*(\Lambda) & 0 \\ 0 & D_R^*(\Lambda) \end{pmatrix} \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^2 D_R^*(\Lambda) \sigma^2 & 0 \\ 0 & -\sigma^2 D_L^*(\Lambda) \sigma^2 \end{pmatrix}.$$

Using the explicit form of $D_{L,R}(\Lambda)$ from Eq. (3.9) together with the properties $\sigma^2 \sigma^i \sigma^2 = -\sigma^{i*}$ and $\sigma^2 \sigma^2 = 1$, it follows that

$$\sigma^2 D_{L,R}^*(\Lambda) \sigma^2 = D(\Lambda)_{R,L} \quad \Rightarrow \quad \gamma^2 D^*(\Lambda) \gamma^2 = -D(\Lambda). \quad (4.37)$$

In terms of Weyl spinors, the charge-conjugate spinor takes the form

$$\psi^c = \begin{pmatrix} \psi_L^c \\ \psi_R^c \end{pmatrix} = \begin{pmatrix} -i\sigma^2 \psi_R^* \\ i\sigma^2 \psi_L^* \end{pmatrix}. \quad (4.38)$$

Let's express ψ^* through the conjugate spinor: $\psi^* = (\psi^\dagger)^T = (\bar{\psi} \gamma^0)^T = \gamma^0 \bar{\psi}^T$. Defining the **charge-conjugation matrix** $C = i\gamma^2 \gamma^0$, we arrive at

$$\psi^c = C^T \bar{\psi}^T, \quad \bar{\psi}^c = \psi^T C^T. \quad (4.39)$$

The transpose on a spinor is not really necessary; it just means that $\psi_\alpha^c = (C^T)_{\alpha\beta} \bar{\psi}_\beta = \bar{\psi}_\beta C_{\beta\alpha}$. The charge-conjugation matrix has some useful properties:

$$C^\dagger = C^T = C^{-1} = -C, \quad C \gamma_5^T C^T = \gamma_5, \quad C \gamma_\mu^T C^T = -\gamma_\mu. \quad (4.40)$$

Since charge conjugation does not change the spacetime argument, we can identify it directly with the operator transformation:

$$U_C \psi U_C^{-1} = C^T \bar{\psi}^T, \quad U_C \bar{\psi} U_C^{-1} = \psi^T C^T. \quad (4.41)$$

If we insert the Fourier decomposition and use the relations $\gamma^2 u_{p,s} = v_{p,s}$ and $\gamma^2 v_{p,s} = u_{p,s}$, which follow again from Eq. (3.59), we arrive at

$$U_C a_{p,s} U_C^{-1} = b_{p,s}, \quad U_C b_{p,s} U_C^{-1} = a_{p,s}. \quad (4.42)$$

As desired, charge conjugation transforms a particle $|\mathbf{p}, s, a\rangle$ into its antiparticle $|\mathbf{p}, s, b\rangle$. Recall that the state $|\mathbf{p}, s, a\rangle$ describes a particle with spin polarization $s/2$ and the state $|\mathbf{p}, s, b\rangle$ an antiparticle with spin polarization $-s/2$; therefore, charge conjugation also reverses the helicity.

Time reversal. Although the time reversal operation $x \rightarrow x' = (-t, \mathbf{x})$ looks similar to the parity transformation, it is probably the most confusing of the discrete symmetries and has a rather special status. In the classical theory, all particles of the time-mirrored system follow their trajectories backwards: the momenta and angular momenta are reversed, and the roles of the initial and final configurations are interchanged. A Dirac spinor transforms as

$$\psi'(x') = \gamma^0 \gamma^5 \psi^c(x) = \gamma^1 \gamma^3 \psi^*(x), \quad (4.43)$$

which can be derived from the transformation behavior of the Dirac equation, or that of fermion bilinears. The need for complex conjugation can be understood intuitively from the picture of antiparticles as particles moving backwards in time (a time reversal of the phase $e^{-iE_p t}$ would lead to negative energies of the mirrored system and necessitates a sign change of i). Correspondingly, the Weyl spinors transform as $\psi'_{L,R}(x') = i\sigma^2 \psi^*_{L,R}(x)$.

The speciality of time reversal is that, when taking matrix elements, it exchanges the in and out states:

$$\langle U_T \lambda_1 | \psi_\alpha(x') | U_T \lambda_2 \rangle = (\gamma^1 \gamma^3)_{\alpha\beta} \langle \lambda_1 | \psi_\beta(x) | \lambda_2 \rangle^* = (\gamma^1 \gamma^3)_{\alpha\beta} \langle \lambda_2 | \psi_\beta^\dagger(x) | \lambda_1 \rangle, \quad (4.44)$$

and therefore we cannot simply compare both sides of the equation anymore to obtain a transformation law for the field operators. To do so, we must identify U_T with an **antiunitary** operator, which leads to

$$U_T \psi(x) U_T^{-1} = \gamma^1 \gamma^3 \psi(x'), \quad U_T \bar{\psi}(x) U_T^{-1} = \bar{\psi}(x') \gamma^3 \gamma^1, \quad (4.45)$$

again ignoring possible phases. This is compatible with the **Wigner theorem**, which states that symmetries in the quantum theory must be implemented by unitary or antiunitary operators. Note that an antiunitary operator induces complex conjugation for numbers: $U_T c U_T^{-1} = c^*$. However, since the transformation of the quantum fields $\psi, \bar{\psi}$ no longer requires complex conjugation, the transformation does not send particles to antiparticles but rather particles to *particles*.

The point is that Hilbert state vectors that differ only by phases are physically equivalent, which is why it is sufficient to demand $|\langle U\lambda_1 | U\lambda_2 \rangle| = |\langle \lambda_1 | \lambda_2 \rangle|$ for symmetry operations. This can be realized by a unitary operator,

$$\langle U\lambda_1 | U\lambda_2 \rangle = \langle \lambda_1 | \lambda_2 \rangle, \quad U(c_1 |\lambda_1\rangle + c_2 |\lambda_2\rangle) = c_1 U |\lambda_1\rangle + c_2 U |\lambda_2\rangle \quad (4.46)$$

or an antiunitary operator:

$$\langle U\lambda_1 | U\lambda_2 \rangle = \langle \lambda_1 | \lambda_2 \rangle^* = \langle \lambda_2 | \lambda_1 \rangle, \quad U(c_1 |\lambda_1\rangle + c_2 |\lambda_2\rangle) = c_1^* U |\lambda_1\rangle + c_2^* U |\lambda_2\rangle. \quad (4.47)$$

Clearly, both possibilities are compatible with the symmetry requirement, but the essence of the Wigner theorem (whose proof is rather lengthy) is that these are the *only* options. Note that in both cases $U^\dagger U = U U^\dagger = 1$, but the definition of the hermitian conjugate changes in the antiunitary case: $\langle \lambda_1 | U^\dagger \lambda_2 \rangle = \langle \lambda_2 | U \lambda_1 \rangle$. Hence, Eq. (4.44) requires U_T to be antiunitary:

$$\langle U_T \lambda_1 | \psi(x') | U_T \lambda_2 \rangle = \langle \lambda_2 | U_T^\dagger \psi(x')^\dagger U_T | \lambda_1 \rangle, \quad (4.48)$$

and the comparison with the r.h.s. leads to Eq. (4.45) (again, up to an irrelevant phase factor).

		C	P	T	CPT
S	$\bar{\psi}\psi$	1	1	1	1
P	$\bar{\psi}i\gamma_5\psi$	1	-1	-1	1
V	$\bar{\psi}\gamma^\mu\psi$	-1	(1, -1)	(1, -1)	-1
A	$\bar{\psi}\gamma^\mu\gamma_5\psi$	1	(-1, 1)	(1, -1)	-1
T	$\bar{\psi}\sigma^{\mu\nu}\psi$	-1	$\left(\begin{array}{c c} 1 & -1 \\ \hline -1 & 1 \end{array}\right)$	$\left(\begin{array}{c c} -1 & 1 \\ \hline 1 & -1 \end{array}\right)$	1
	∂_μ	1	(1, -1)	(-1, 1)	-1

TABLE 1: Transformation properties under C , P and T .

CPT. The transformation properties of the various fermion bilinears : $\bar{\psi}(x)\Gamma\psi(x)$: under C , P and T are summarized in Table 1. The free Dirac action is invariant under C , P and T separately. We can construct more general actions that violate any of these symmetries, but since they must be Lorentz scalars, the free Lorentz indices in γ^μ , $\gamma^\mu\gamma_5$ and $\sigma^{\mu\nu}$ must be contracted with the derivative ∂_μ (or other bilinears). As a consequence, the combined symmetry CPT is always conserved: one cannot build a Lorentz-invariant quantum field theory with a hermitian Hamiltonian that violates CPT .

For example, under charge conjugation the bilinears behave as

$$\begin{aligned} \bar{\psi}\Gamma\psi &\rightarrow U_C(\bar{\psi}\Gamma\psi)U_C^{-1} = \psi^T C^T \Gamma C^T \bar{\psi}^T = (\psi)_\alpha (C^T \Gamma C^T)_{\alpha\beta} \bar{\psi}_\beta \\ &= -\bar{\psi}_\beta (C^T \Gamma C^T)_{\alpha\beta} (\psi)_\alpha = \bar{\psi} (C \Gamma^T C^T) \psi, \end{aligned} \quad (4.49)$$

where we used fermion anticommutation: $\psi_\alpha \bar{\psi}_\alpha = -\bar{\psi}_\alpha \psi_\alpha$ (the infinite constant vanishes by normal ordering). Together with the relations (4.40) it is then straightforward to obtain the ‘ C ’ column in Table 1; note that the vector and tensor bilinears switch sign under charge conjugation. Similarly, under parity one has

$$\bar{\psi}(x)\Gamma\psi(x) \rightarrow U_P(\bar{\psi}(x)\Gamma\psi(x))U_P^{-1} = \bar{\psi}(x')(\gamma^0\Gamma\gamma^0)\psi(x'), \quad (4.50)$$

and time reversal leads to

$$\bar{\psi}(x)\Gamma\psi(x) \rightarrow U_T(\bar{\psi}(x)\Gamma\psi(x))U_T^{-1} = \bar{\psi}(x')(\gamma^3\gamma^1\Gamma^*\gamma^1\gamma^3)\psi(x'), \quad (4.51)$$

where the complex conjugate Γ^* is a consequence of the antiunitarity: $U_T\Gamma U_T^{-1} = \Gamma^*$. Since we can express time reversal through charge conjugation via Eq. (4.43), the result can be also written as

$$U_T(\bar{\psi}(x)\Gamma\psi(x))U_T^{-1} = \bar{\psi}(x')\gamma_5\gamma^0(C\Gamma^T C^T)\gamma^0\gamma_5\psi(x'). \quad (4.52)$$

In summary, the signs in the table are simply obtained from the signs of

$$C \rightarrow C\Gamma^T C^T, \quad P \rightarrow \gamma^0\Gamma\gamma^0, \quad T \rightarrow \gamma_5\gamma^0(C\Gamma^T C^T)\gamma^0\gamma_5. \quad (4.53)$$

Taking everything in combination, the CPT symmetry amounts to $\gamma_5\Gamma\gamma_5$ together with a spacetime reflection $x \rightarrow -x$.

5 Electromagnetic field

Classical electromagnetism. Classical Maxwell equations:

$$\nabla \cdot \mathbf{E} = \rho, \quad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0. \quad (5.1)$$

The inhomogeneous equations imply local charge conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \quad (5.2)$$

To arrive at covariant equations, define the current $j^\mu = (\rho, \mathbf{j})$ and the antisymmetric field-strength tensor $F^{\mu\nu} = -F^{\nu\mu}$ as

$$F^{ij} = -\varepsilon_{ijk} B^k \quad \Leftrightarrow \quad B^i = -\frac{1}{2} \varepsilon_{ijk} F^{jk}, \quad F^{0i} = -E^i, \quad (5.3)$$

together with its dual:

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \quad \Rightarrow \quad \begin{aligned} \tilde{F}^{ij} &= \varepsilon_{ijk} F^{0k} = -\varepsilon_{ijk} E^k, \\ \tilde{F}^{0i} &= -\frac{1}{2} \varepsilon_{ijk} F^{jk} = B^i. \end{aligned} \quad (5.4)$$

The combination of Maxwell equations and current conservation becomes

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad \partial_\mu \tilde{F}^{\mu\nu} = 0, \quad \partial_\mu j^\mu = 0. \quad (5.5)$$

Current conservation follows again from the inhomogeneous Maxwell equation because $\partial_\mu \partial_\nu F^{\mu\nu} = 0$. The homogeneous Maxwell equations allow us to construct a vector potential $A^\mu = (\phi, \mathbf{A})$ via

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad \Leftrightarrow \quad \mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad (5.6)$$

which is then only determined up to a derivative:

$$A'^\mu(x) = A^\mu(x) + \partial^\mu \varepsilon(x) \quad \Leftrightarrow \quad \begin{aligned} \phi' &= \phi + \frac{\partial \varepsilon}{\partial t}, \\ \mathbf{A}' &= \mathbf{A} - \nabla \varepsilon. \end{aligned} \quad (5.7)$$

In other words, $F^{\mu\nu}$ and therefore the fields \mathbf{E} and \mathbf{B} are invariant under **local gauge transformations**, and vector fields A^μ that differ only by such a term are physically equivalent. Local gauge invariance will eventually become the fundamental construction principle for interacting field theories. At the present stage it merely corresponds to a redundancy in the description of the system, and to determine the true physical degrees of freedom we must be sure to divide out this redundancy (which will be the main difficulty in quantizing the system). In summary, all three equations in Eq. (5.5) can be combined into the Maxwell equations

$$\square A^\mu - \partial^\mu \partial_\nu A^\nu = j^\mu. \quad (5.8)$$

Lagrangian of the electromagnetic field. We interpret the vector field $A^\mu(x)$ now as the fundamental electromagnetic field. The Maxwell equations follow as the equations of motion from the action

$$S = \int d^4x \mathcal{L} = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu \right] = \int d^4x \left[\frac{1}{2} (\mathbf{E}^2 - \mathbf{B}^2) - j^\mu A_\mu \right]. \quad (5.9)$$

The current $j^\mu(x)$ that appears here as a static source term is presently just a compromise that we will eventually get rid of: in a fully interacting theory the current will emerge from other fields and thereby carry their dynamical information (in a free field theory $j^\mu = 0$). Let's rewrite the action in terms of A^μ and its derivatives:

$$\begin{aligned} S &= \int d^4x \left[-\frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu) \partial^\mu A^\nu - j^\mu A_\mu \right] \\ &\stackrel{\text{p.I.}}{=} \int d^4x \left[\frac{1}{2} A_\mu (\square g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu - j^\mu A_\mu \right]. \end{aligned} \quad (5.10)$$

From the first line above it is easy to derive the Maxwell equations via

$$\frac{\partial \mathcal{L}}{\partial A^\nu} = -j_\nu, \quad \frac{\partial \mathcal{L}}{\partial (\partial_\mu A^\nu)} = -\partial^\mu A_\nu + \partial_\nu A^\mu = -F^\mu{}_\nu, \quad (5.11)$$

and we can read off the canonical conjugate momentum:

$$\Pi_\nu = \frac{\partial \mathcal{L}}{\partial (\partial_0 A^\nu)} = -F^0{}_\nu \quad \Rightarrow \quad \Pi^0 = 0, \quad \mathbf{\Pi} = \mathbf{E}. \quad (5.12)$$

Note that the time component A_0 has no conjugate momentum, which will produce difficulties in the quantization. The Hamilton function becomes

$$\begin{aligned} H &= \int d^3x \mathcal{H} = \int d^3x \left[\Pi_\nu \frac{\partial A^\nu}{\partial t} - \mathcal{L} \right] \\ &= \int d^3x \left[\frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2) + \mathbf{E} \cdot \nabla \phi + \rho \phi - \mathbf{j} \cdot \mathbf{A} \right] \\ &\stackrel{\text{p.I.}}{=} \int d^3x \left[\frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2) - \mathbf{j} \cdot \mathbf{A} \right]. \end{aligned} \quad (5.13)$$

Poincaré transformations. Let's study the conservation laws that follow from the Poincaré invariance of the action. According to Eq. (1.40), the generic infinitesimal current takes the form

$$-\delta j^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu A^\nu)} \delta A^\nu - T^{\mu\nu} \delta x_\nu = -F^{\mu\nu} \delta A_\nu - T^{\mu\nu} \delta x_\nu, \quad (5.14)$$

and the energy-momentum tensor is given by

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu A^\alpha)} \partial^\nu A^\alpha - g^{\mu\nu} \mathcal{L} = -F^{\mu\alpha} \partial_\nu A^\alpha - g^{\mu\nu} \mathcal{L}. \quad (5.15)$$

Translation invariance ($\delta A^\nu = 0$, $\delta x_\nu = a_\nu$) implies that it is conserved; however, in the presence of the current $j^\mu(x)$ its divergence is $\partial_\mu T^{\mu\nu} = (\partial^\nu j_\alpha) A^\alpha \neq 0$. This is

just because we treat the current as an external source: in a complete theory it would emerge from other fields which also contribute to the energy-momentum tensor. (By the way, note that δj^μ has nothing to do with j^μ .)

Under Lorentz transformations the field transforms as

$$A'^\mu(x') = \Lambda^\mu_\nu A^\nu \quad \Leftrightarrow \quad \begin{aligned} \delta x_\alpha &= \varepsilon_{\alpha\beta} x^\beta, \\ \delta A_\alpha &= \varepsilon_{\alpha\beta} A^\beta = \frac{i}{2} \varepsilon_{\mu\nu} (M^{\mu\nu})_{\alpha\beta} A^\beta, \end{aligned} \quad (5.16)$$

because in the vector representation the irreducible representation matrix is just the Lorentz transformation itself. The infinitesimal generator of Lorentz transformations was given in Eq. (2.54):

$$(M^{\mu\nu})_{\alpha\beta} = -i(\delta_\alpha^\mu \delta_\beta^\nu - \delta_\alpha^\nu \delta_\beta^\mu). \quad (5.17)$$

The corresponding infinitesimal current defines the angular momentum density,

$$-\delta j^\mu = -\frac{1}{2} \varepsilon_{\alpha\beta} \underbrace{(F^{\mu\alpha} A^\beta - F^{\mu\beta} A^\alpha + T^{\mu\alpha} x^\beta - T^{\mu\beta} x^\alpha)}_{=: m^{\mu,\alpha\beta}}, \quad (5.18)$$

which is conserved (if the external current $j^\mu = 0$): $\partial_\mu m^{\mu,\alpha\beta} = 0$. Once again we can insert the explicit form of the energy-momentum tensor and isolate the orbital angular-momentum part:

$$T^{\mu\alpha} x^\beta - T^{\mu\beta} x^\alpha = iF^{\mu\rho} L^{\alpha\beta} A_\rho + (x^\alpha g^{\mu\beta} - x^\beta g^{\mu\alpha}) \mathcal{L}, \quad (5.19)$$

where $L^{\alpha\beta}$ was defined in Eq. (1.44). In combination with the spin contribution, the angular momentum density becomes

$$m^{\mu,\alpha\beta} = iF^{\mu\rho} [(M^{\alpha\beta})_{\rho\sigma} + g_{\rho\sigma} L^{\alpha\beta}] A^\sigma =: iF^{\mu\rho} (J^{\alpha\beta})_{\rho\sigma} A^\sigma. \quad (5.20)$$

Hence, the charge that is conserved under rotations is the angular momentum of the electromagnetic field

$$\int d^3x m^{0,ij} = -i \int d^3x E^k (J^{ij})_{kl} A^l =: -\varepsilon_{ijk} \tilde{\mathcal{J}}^k, \quad (5.21)$$

whose explicit form is

$$\tilde{\mathcal{J}} = \int d^3x (\mathbf{E} \times \mathbf{A} - iE^k \mathbf{L} A^k) \quad (5.22)$$

with $\mathbf{L} = \mathbf{x} \times (-i\nabla)$. The spin of the electromagnetic field is $\int d^3x \mathbf{E} \times \mathbf{A}$.

The energy-momentum tensor $T^{\mu\nu}$ in Eq. (5.15) is neither symmetric in its indices nor gauge-invariant (because it depends explicitly on A^α). An alternative *symmetric* form of the energy-momentum tensor is the **Belinfante tensor**, which is still conserved and therefore physically equivalent: (Ex)

$$\Theta^{\alpha\beta} = T^{\alpha\beta} - \frac{1}{2} \partial_\mu (s^{\mu,\alpha\beta} + s^{\alpha,\beta\mu} - s^{\beta,\mu\alpha}). \quad (5.23)$$

Here, $s^{\mu,\alpha\beta}$ is the spin contribution to the angular momentum density, i.e.

$$m^{\mu,\alpha\beta} = s^{\mu,\alpha\beta} + T^{\mu\alpha} x^\beta - T^{\mu\beta} x^\alpha. \quad (5.24)$$

This statement is general and holds independently of the nature of the fields. Its proof is simple: by construction, $s^{\mu,\alpha\beta}$ is antisymmetric in α, β and therefore

$$\Theta^{\alpha\beta} - \Theta^{\beta\alpha} = T^{\alpha\beta} - T^{\beta\alpha} - \partial_\mu s^{\mu,\alpha\beta}. \quad (5.25)$$

On the other hand, by taking the derivative of $m^{\mu,\alpha\beta}$ we see that

$$\partial_\mu m^{\mu,\alpha\beta} = \partial_\mu s^{\mu,\alpha\beta} - (T^{\alpha\beta} - T^{\beta\alpha}) = 0 \quad (5.26)$$

and therefore $\Theta^{\alpha\beta}$ is symmetric. (We used the fact that $T^{\alpha\beta}$ and $m^{\mu,\alpha\beta}$ are conserved.) $\Theta^{\alpha\beta}$ is conserved because the bracket in Eq. (5.23) is antisymmetric under an exchange $\mu \leftrightarrow \alpha$:

$$\partial_\alpha \Theta^{\alpha\beta} = -\frac{1}{2} \partial_\alpha \partial_\mu (s^{\mu,\alpha\beta} - s^{\alpha,\mu\beta} - s^{\beta,\mu\alpha}) = 0. \quad (5.27)$$

Let's work out the Belinfante tensor for the electromagnetic field. When inserting $s^{\mu,\alpha\beta} = F^{\mu\alpha} A^\beta - F^{\mu\beta} A^\alpha$ into Eq. (5.23) we obtain

$$\begin{aligned} \Theta^{\alpha\beta} &= T^{\alpha\beta} - \partial_\mu (F^{\mu\alpha} A^\beta) = -F^{\alpha\mu} \partial^\beta A_\mu - \partial_\mu (F^{\mu\alpha} A^\beta) - g^{\alpha\beta} \mathcal{L} \\ &= F^{\alpha\mu} F_\mu{}^\beta - j^\alpha A^\beta - g^{\alpha\beta} \mathcal{L}. \end{aligned} \quad (5.28)$$

Apart from the $j \cdot A$ term it is now also gauge-invariant. (Although $T^{\mu\nu}$ was gauge dependent, the charges derived from it are gauge-invariant because gauge transformations would only produce surface terms – see Maggiore, p.68.) Its components are

$$\begin{aligned} \Theta^{00} &= (F^{0i})^2 - j^0 A^0 - \mathcal{L} = \frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2) - \mathbf{j} \cdot \mathbf{A} = \mathcal{H}, \\ \Theta^{0i} &= F^{0k} F^{ik} - j^0 A^i = (\mathbf{E} \times \mathbf{B})_i - \rho \mathbf{A}. \end{aligned} \quad (5.29)$$

Likewise, Θ^{ij} would give the Maxwell stress tensor. The corresponding charges are the components of the four momentum $P^\mu = \int d^3x \Theta^{0i}$. Therefore, in the absence of an external current j^μ , the energy density of the electromagnetic field is $\frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2)$, its momentum density is the Poynting vector $\mathbf{E} \times \mathbf{B}$, and its spin density is $\mathbf{E} \times \mathbf{A}$.

Gauge fixing. Gauge invariance poses new problems for the quantization of the electromagnetic field. The field carries spin 1 and is of bosonic nature, so in principle we should impose the commutator relations

$$[A^\mu(x), \Pi^\nu(y)]_{x^0=y^0} = i g^{\mu\nu} \delta^3(\mathbf{x} - \mathbf{y}). \quad (5.30)$$

Unfortunately this gives a contradiction because $\Pi^0 = 0$ vanishes and cannot have a non-trivial commutator with A^0 . This reflects the redundancy that is inherent in the field A^μ . Gauge invariance tells us that we should restrict ourselves to a subset of fields A^μ that satisfy a certain gauge-fixing condition, for example

- the **Lorenz gauge** $\partial_\mu A^\mu = 0$: it only partially fixes the gauge, because we are still free to perform a *residual* gauge transformation $A'^\mu = A^\mu + \partial^\mu \varepsilon$ as long as $\square \varepsilon = 0$. The Maxwell equations in the Lorenz gauge simply become $\square A^\mu = j^\mu$.
- the **radiation gauge** $A^0 = 0, \nabla \cdot \mathbf{A} = 0$: here the gauge fixing is complete because the remaining gauge parameter ε can be only a constant. The radiation gauge implies the Lorenz gauge $\partial_\mu A^\mu = 0$ but it is more restrictive.

There are two possible strategies for quantizing the theory: we could either fix the gauge in advance and thereby eliminate the unphysical degrees of freedom. The price we have to pay is the loss of manifest Lorentz covariance, and we have to check at the end of the quantization procedure that Lorentz symmetry is still intact. The second

option is to work with the full gauge field A^μ and start from a *modified* Lagrangian where Π^0 does not vanish. This will introduce spurious degrees of freedom which we have to eliminate at the end.

We follow the second avenue and start from the following ‘gauge-fixed’ Lagrangian, where the gauge-fixing condition $\partial_\mu A^\mu = 0$ is implemented in the form of a Lagrange multiplier:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{2} (\partial \cdot A)^2. \quad (5.31)$$

The resulting equations of motion become

$$\partial_\mu F^{\mu\nu} + \lambda \partial^\nu \partial_\mu A^\mu = 0 \quad \Leftrightarrow \quad \square A^\mu + (\lambda - 1) \partial^\mu \partial_\nu A^\nu = 0. \quad (5.32)$$

Taking their divergence yields $\lambda \square \partial_\mu A^\mu = 0$, which means that $\partial_\mu A^\mu$ must be a free scalar field that satisfies the massless Klein-Gordon equation. The additional term in the Lagrangian ensures that $\Pi^0 = -\lambda \partial_\mu A^\mu$ is no longer zero (the spatial components Π^i are unchanged), so in principle we can proceed with the quantization. Although we could discuss what follows for general λ , we set $\lambda = 1$ (**Feynman gauge**) because this simulates the Lorenz gauge condition in the Maxwell equations: $\square A^\mu = 0$. (The limit where $\lambda \rightarrow \infty$ at the end of all calculations is called Landau gauge.) Following the steps in Eq. (5.10), it is easy to show that the action obtained from the Lagrangian (5.31) with $\lambda = 1$ is equivalent to that of the **Fermi Lagrangian**

$$\mathcal{L} = -\frac{1}{2} (\partial_\mu A_\nu) (\partial^\mu A^\nu). \quad (5.33)$$

Its canonical conjugate momentum is

$$\Pi_\nu = \frac{\partial \mathcal{L}}{\partial (\partial_0 A^\nu)} = -\partial_0 A_\nu = -\dot{A}_\nu, \quad (5.34)$$

and the Hamiltonian of this theory becomes

$$H = \int d^3x \left(\Pi_\nu \dot{A}^\nu - \mathcal{L} \right) = \int d^3x \left[-\frac{1}{2} \dot{A}^2 - \frac{1}{2} (\nabla A_\nu) (\nabla A^\nu) \right]. \quad (5.35)$$

Polarization vectors. The solutions of the free Maxwell equations have the form

$$A^\mu(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} \sum_{\lambda=0}^3 \left(a_{p,\lambda} \epsilon_{p,\lambda}^\mu e^{-ipx} + a_{p,\lambda}^\dagger \epsilon_{p,\lambda}^{*\mu} e^{ipx} \right), \quad (5.36)$$

which is compatible with $\square A^\mu = 0$ as an operator equation as long as the four-vector p^μ is lightlike: $p^2 = 0 \Leftrightarrow E_p = |\mathbf{p}|$. The 4 linearly independent polarization vectors $\epsilon_{p,\lambda}^\mu = \epsilon^\mu(\mathbf{p}, \lambda)$ depend on p^μ , and they can always be chosen to satisfy the following orthogonality and completeness relations:

$$\epsilon_{p,\lambda} \cdot \epsilon_{p,\lambda'} = g_{\lambda\lambda'}, \quad \sum_{\lambda\lambda'} g^{\lambda\lambda'} \epsilon_{p,\lambda}^\mu \epsilon_{p,\lambda'}^\nu = g^{\mu\nu}. \quad (5.37)$$

The first relation implies that $\epsilon_{p,0}^2 = 1$ and $\epsilon_{p,i}^2 = -1$ so that $\epsilon_{p,0}^\mu$ is timelike whereas the others with $i = 1, 2, 3$ are spacelike. In particular, without going into a specific reference frame one can proceed as follows:

- The most general timelike vector that satisfies $n^2 = 1$ can be written in the form $n = (\sqrt{1 + \mathbf{n}^2}, \mathbf{n})^T$. Therefore, set $\varepsilon_{p,0}^\mu = n^\mu$ for the **timelike** polarization. The remaining $\varepsilon_{p,i}^\mu$ must be transverse to n^μ with $n \cdot \varepsilon_{p,i} = 0$.
- Choose $\varepsilon_{p,1}^\mu$ and $\varepsilon_{p,2}^\mu$ **transverse** to p^μ , so that $p \cdot \varepsilon_{p,i} = 0$ for $i = 1, 2$.
- The remaining polarization vector $\varepsilon_{p,3}^\mu$ must be a linear combination of n^μ and p^μ . The conditions $n \cdot \varepsilon_{p,3} = 0$ and $\varepsilon_{p,3}^2 = -1$ fix it uniquely: $\varepsilon_{p,3}^\mu = p^\mu / (p \cdot n) - n^\mu$. We call it the **longitudinal** polarization.

For example, with \mathbf{p} in z -direction and $\mathbf{n} = 0$ this implies

$$p = |\mathbf{p}| \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad n = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \varepsilon_{p,0\dots 3}^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (5.38)$$

Canonical quantization. Expressed in terms of the conjugate momentum (5.34), the commutation relations (5.30) take the form

$$\begin{aligned} [A^\mu(x), \dot{A}^\nu(x)]_{x^0=y^0} &= -i g^{\mu\nu} \delta^3(\mathbf{x} - \mathbf{y}), \\ [A^\mu(x), A^\nu(y)]_{x^0=y^0} &= 0, \\ [\dot{A}^\mu(x), \dot{A}^\nu(y)]_{x^0=y^0} &= 0. \end{aligned} \quad (5.39)$$

Note that the spatial components behave like ordinary scalar fields with respect to the commutator relation, whereas the sign for the timelike component is reversed. To extract the commutation relations for the ladder operators we can simply copy the steps from Eqs. (2.6–2.11) for the scalar field; the result is

$$[a_{p,\lambda}, a_{p',\lambda'}^\dagger] = -2E_p g_{\lambda\lambda'} \delta^3(\mathbf{p} - \mathbf{p}') \quad (5.40)$$

with all other commutators zero. Likewise, the momentum operator turns out to be

$$\begin{aligned} P^\mu &= - \int \frac{d^3p}{2E_p} p^\mu \sum_{\lambda\lambda'} g^{\lambda\lambda'} a_{p,\lambda}^\dagger a_{p,\lambda'} \Big|_{p^0=E_p=|\mathbf{p}|} \\ &= \int \frac{d^3p}{2E_p} p^\mu \left[- a_{p,0}^\dagger a_{p,0} + \sum_{\lambda=1}^3 a_{p,\lambda}^\dagger a_{p,\lambda} \right]_{p^0=E_p=|\mathbf{p}|}. \end{aligned} \quad (5.41)$$

Also here the spatial modes have a positive sign but the timelike component comes with a minus. The number operator has an analogous form,

$$\hat{N} = \int \frac{d^3p}{2E_p} \left[- a_{p,0}^\dagger a_{p,0} + \sum_{\lambda=1}^3 a_{p,\lambda}^\dagger a_{p,\lambda} \right]_{p^0=E_p=|\mathbf{p}|}. \quad (5.42)$$

Despite appearances, the minus sign does not imply negative eigenvalues for these operators because when they act on a state $a_{k,0}^\dagger |0\rangle$ the sign cancels with that in the commutator relation:

$$P^\mu a_{k,0}^\dagger |0\rangle = k^\mu a_{k,0}^\dagger |0\rangle, \quad \hat{N} a_{k,0}^\dagger |0\rangle = a_{k,0}^\dagger |0\rangle. \quad (5.43)$$

But this is exactly the problem: the one-particle states $a_{k,0}^\dagger |0\rangle$ with timelike polarization $\lambda = 0$ have a *negative norm*,

$$\langle 0 | a_{q,0} a_{k,0}^\dagger | 0 \rangle = -2E_k \delta^3(\mathbf{k} - \mathbf{q}), \quad (5.44)$$

which spoils the unitarity of the theory. How can we resolve this?

Gupta-Bleuler method. So far our quantization procedure is incomplete because we have not yet implemented the constraint $\partial_\mu A^\mu = 0$. It is impossible to impose it as an operator equation for the fields, because this would contradict our commutator relations:

$$0 \stackrel{!}{=} [A^\mu(x), \partial_\nu A^\nu(y)]_{x^0=y^0} = [A^\mu(x), \dot{A}^0(y)]_{x^0=y^0} = -ig^{\mu 0} \delta^3(\mathbf{x} - \mathbf{y}) \neq 0. \quad (5.45)$$

What we can do instead is to implement it not at the level of the fields, but rather as a restriction on the Hilbert space. Let's decompose the field $A^\mu(x)$ into positive- and negative-frequency modes

$$\begin{aligned} A_+^\mu(x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} \sum_{\lambda=0}^3 a_{p,\lambda} \epsilon_{p,\lambda}^\mu e^{-ipx}, \\ A_-^\mu(x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} \sum_{\lambda=0}^3 a_{p,\lambda}^\dagger \epsilon_{p,\lambda}^{*\mu} e^{ipx}, \end{aligned} \quad (5.46)$$

so that $A^\mu = A_+^\mu + A_-^\mu$. We say that the **physical states** $|\psi\rangle \in \mathbb{H}_{\text{phys}}$ are those states that satisfy the Gupta-Bleuler condition

$$\partial \cdot A_+ |\psi\rangle \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \langle \psi | \partial \cdot A_- = 0. \quad (5.47)$$

The two conditions are equivalent because $A_+^\dagger = A_-$, and taken together they imply that the classical constraint $\partial \cdot A = 0$ is now realized in the form of an expectation value:

$$\langle \psi | \partial \cdot A | \psi \rangle = \langle \psi | \partial \cdot A_+ + \partial \cdot A_- | \psi \rangle = 0. \quad (5.48)$$

We can work out the consequences of this relation by writing $\partial \cdot A_+$ in Fourier modes. According to our construction of the polarization vectors, their contraction with the lightlike momentum p^μ gives

$$p_\mu \epsilon_{p,\lambda}^\mu = \begin{cases} p \cdot n & \lambda = 0 \\ 0 & \lambda = 1, 2 \\ -p \cdot n & \lambda = 3, \end{cases} \quad (5.49)$$

and therefore

$$\partial \cdot A_+ = -i \int \frac{d^3p}{2E_p} e^{-ipx} \sum_\lambda a_{p,\lambda} p_\mu \epsilon_{p,\lambda}^\mu = -i \int \frac{d^3p}{2E_p} e^{-ipx} p \cdot n (a_{p,0} - a_{p,3}). \quad (5.50)$$

Hence, the condition (5.47) for physical states $|\psi\rangle$ is equivalent to the condition

$$a_{p,0} |\psi\rangle \stackrel{!}{=} a_{p,3} |\psi\rangle. \quad (5.51)$$

Now observe that whenever we evaluate expectation values of operators of the form (5.41) or (5.42), we arrive at

$$\langle \psi | a_{p,0}^\dagger a_{p,0} - a_{p,3}^\dagger a_{p,3} | \psi \rangle = \langle \psi | (a_{p,0}^\dagger - a_{p,3}^\dagger) a_{p,3} | \psi \rangle = 0. \quad (5.52)$$

Therefore, the timelike and longitudinal photons cancel each other in matrix elements, and only the transverse, physical polarizations $\lambda = 1, 2$ survive:

$$\langle \psi | \left[-a_{p,0}^\dagger a_{p,0} + \sum_{\lambda=1}^3 a_{p,\lambda}^\dagger a_{p,\lambda} \right] | \psi \rangle = \langle \psi | \sum_{\lambda=1}^2 a_{p,\lambda}^\dagger a_{p,\lambda} | \psi \rangle. \quad (5.53)$$

Physical state space. Let's find out what this means for a 'physical' one-particle state. We start by writing it as the most general superposition of polarization states with momentum \mathbf{k} :

$$|\psi\rangle = \sum_{\lambda} c_{\lambda} a_{\mathbf{k},\lambda}^\dagger |0\rangle. \quad (5.54)$$

Applying the condition (5.51) to it entails

$$(a_{p,0} - a_{p,3}) |\psi\rangle = \sum_{\lambda} c_{\lambda} \underbrace{(a_{p,0} - a_{p,3}) a_{\mathbf{k},\lambda}^\dagger |0\rangle}_{-2E_p \delta^3(\mathbf{p}-\mathbf{k}) (g_{\lambda 0} - g_{\lambda 3}) |0\rangle} \stackrel{!}{=} 0, \quad (5.55)$$

and therefore $c_0 = -c_3$, whereas c_1 and c_2 are unconstrained. This means there are two types of 'physical states' $|\psi\rangle$ that satisfy the transversality condition:

$$|\psi_T\rangle = (c_1 a_{\mathbf{k},1}^\dagger + c_2 a_{\mathbf{k},2}^\dagger) |0\rangle, \quad |\phi\rangle = (a_{\mathbf{k},0}^\dagger - a_{\mathbf{k},3}^\dagger) |0\rangle, \quad (5.56)$$

whereas the negative-norm state $a_{\mathbf{k},0}^\dagger |0\rangle$ does *not* satisfy the constraint. On the other hand, a massless photon has only two physical polarizations, so what is the meaning of the state $|\phi\rangle$? Consider the scalar product

$$\langle \psi | \phi \rangle = \langle \psi | (a_{\mathbf{k},0}^\dagger - a_{\mathbf{k},3}^\dagger) |0\rangle = 0, \quad (5.57)$$

which must be zero because of Eq. (5.51). Since this holds for all states $|\psi\rangle$, and $|\phi\rangle$ is also one of them, it implies in particular $\langle \phi | \phi \rangle = 0$, i.e., the state $|\phi\rangle$ has zero norm. Because $|\phi\rangle$ is orthogonal to all $|\psi\rangle$, all scalar products of a general state $|\psi_T\rangle + c |\phi\rangle$ with any other physical state are the same as those with $|\psi_T\rangle$ alone, and therefore $|\phi\rangle$ decouples from all physical processes. In particular, it does not contribute to any matrix elements such as in Eq. (5.53), which are obtained from the transverse states $|\psi_T\rangle$ only: $\langle \phi | \mathcal{O} | \phi \rangle = 0$. States that decouple from the physics are also called **spurious**.

The decoupling statement will become nontrivial in the presence of interactions. As long as the interactions satisfy gauge invariance, the spurious states decouple from the S -matrix in all *external* legs where only the two transverse polarizations survive (this is a consequence of the Ward identities). However, the spurious states still contribute *internally* in the sense of virtual particles, where they are necessary to preserve unitarity.

6 Interactions and the S-matrix

So far we have been dealing with free, non-interacting quantum field theories for spin-0, spin- $\frac{1}{2}$ and spin-1 particles:

$$\mathcal{L}_0^{\text{sc}} = \frac{1}{2} (\partial\Phi)^2 - \frac{1}{2} m_0^2 \Phi^2, \quad \mathcal{L}_0^{\text{Dirac}} = \bar{\psi} (i\cancel{\partial} - m_0) \psi, \quad \mathcal{L}_0^{\text{em}} = -\frac{1}{4} F^2. \quad (6.1)$$

We denote the free Lagrangian by \mathcal{L}_0 and the mass parameter in the Lagrangian by m_0 . We can solve the corresponding equations of motion (the Klein-Gordon, Dirac and Maxwell equations) exactly in terms of superpositions of plane waves. After quantizing such a theory, the Hilbert space is the Fock space of the multiparticle states that are created from the free vacuum $|0\rangle$.

Interactions. What happens when we include interactions? Let's write the interacting Lagrangian as $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$ and the interacting Hamiltonian as $H = H_0 + H_{\text{int}}$. Examples for interactions are:

- Higher-order terms in theories with one type of field, for example the Φ^3 and Φ^4 interactions in a scalar theory: $\mathcal{L}_{\text{int}} = -\frac{g}{3!} \Phi^3$ or $\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!} \Phi^4$. They describe self-interactions of a scalar particle with respective coupling strengths g and λ .
- Interactions that couple different types of fields, for example the Lagrangian of QED: $\mathcal{L}_{\text{QED}} = \mathcal{L}_0^{\text{Dirac}} + \mathcal{L}_0^{\text{em}} + g \bar{\psi} \cancel{A} \psi$.

Later we will see that the possible forms of interactions are tightly constrained by the requirements of gauge invariance and renormalizability.

To keep the discussion generic, let's stick with scalar fields and work out the consequences of their interactions. Unfortunately this complicates matters enormously. Usually we can no longer solve the equations of motion exactly; for example, the Klein-Gordon equation with Φ^3 and Φ^4 interactions becomes

$$(\square + m_0^2) \Phi = -\frac{g}{2} \Phi^2 - \frac{\lambda}{3!} \Phi^3, \quad (6.2)$$

which is non-linear in the fields. Since the field is not free, there is no simple expansion in terms of creation and annihilation operators. At some given time t_0 , we could try to expand $\Phi(\mathbf{x}, t_0)$ into Fourier modes and formally evolve it with $\Phi(\mathbf{x}, t) = e^{iH(t-t_0)} \Phi(\mathbf{x}, t_0) e^{-iH(t-t_0)}$, but H depends on higher powers of Φ which complicates the solution. A state $a_k^\dagger |0\rangle$ can evolve into $a^{\dagger 2} |0\rangle$, $a^\dagger a^2 |0\rangle$, $a^{\dagger 3} |0\rangle$ terms etc., which would describe the decay of a one-particle state into two- and three-particle states. Hence, $\Phi(x)$ no longer creates just one-particle states but also **multiparticle states**. Similarly, a fermion operator $\bar{\psi}$ in QED would not only create a single electron but also states that contain an electron plus arbitrarily many photons; A^μ would create states that contain besides a single photon also e^+e^- pairs.

As a consequence, the Hilbert space differs from the free theory: the ground state of the free Hamiltonian H_0 was the free vacuum $|0\rangle$; the ground state of the full Hamiltonian is the interacting vacuum $|\Omega\rangle$. The masses m of the 1-particle momentum eigenstates of H no longer equal the mass parameter m_0 in the Lagrangian. The states interact, and there may be bound states.

The basic quantity of interest is then the **scattering amplitude** or transition amplitude between such multiparticle states. Ideally one would like to find the exact solution of the interacting QFT, compute the exact spectrum and calculate the interactions exactly. Unfortunately such analytic solutions are available only for a few special cases. In general one has to resort to numerical methods (lattice QFT, Dyson-Schwinger equations, functional renormalization-group equations, ...) or simplified models. On the other hand, as long as the couplings are small ($g, \lambda \ll 1$), one can view \mathcal{L}_{int} as a small perturbation and expand scattering amplitudes in powers of the coupling constant(s). The resulting **perturbation theory** still allows us to perform analytic calculations and it will be our tool of choice in practice. However, before getting there (in Sec. 7), let us first make some general statements that are also valid non-perturbatively.

Källén-Lehmann spectral representation. How can we determine the masses in an interacting quantum field theory? First of all, Lorentz invariance tells us that the commutation relation $[P^\mu, P^\nu] = 0$ must still hold, which implies that the momentum operator commutes with the Hamiltonian and they are simultaneously diagonalizable: $[H, \mathbf{P}] = 0$. We label their eigenstates by

$$H |\lambda_{\mathbf{p}}\rangle = E_p(\lambda) |\lambda_{\mathbf{p}}\rangle, \quad \mathbf{P} |\lambda_{\mathbf{p}}\rangle = \mathbf{p} |\lambda_{\mathbf{p}}\rangle. \quad (6.3)$$

There are now several types of possible Fock states:

- The ground state or vacuum $|\Omega\rangle$, which is invariant under Poincaré transformations. In particular, this means it has zero energy and momentum: $P^\mu |\Omega\rangle = 0$.
- One-particle states $|\mathbf{p}\rangle$ with momentum \mathbf{p} and energy $E_p = \sqrt{\mathbf{p}^2 + m^2}$, where $m \neq m_0$ is no longer the mass parameter in the Lagrangian.
- N -particle states that are specified by a center-of-mass momentum \mathbf{p} , the relative momenta among the particles, and potentially further parameters. For example, the lowest possible energy of a two-particle state in its rest frame ($\mathbf{p} = 0$) is $2m$, but since the two particles can have relative momentum, which contributes to their total energy, the state can have *any* energy above $2m$. Therefore, the multiparticle states form a continuum. We write the energy of an N -particle state $|\lambda_{\mathbf{p}}\rangle$ as $E_p(\lambda) = (\mathbf{p}^2 + m_\lambda^2)^{1/2}$, where $m_\lambda \geq 2m$ is the invariant mass of the state (its energy in the rest frame).
- Bound states with mass $< 2m$, which have no analogue in the free theory.

The resulting eigenvalue spectrum of H will generally have the form shown in Fig. 6.1. We can then write the completeness relation for the entire Fock space as

$$\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_{\lambda} \int \frac{d^3p}{2E_p(\lambda)} |\lambda_{\mathbf{p}}\rangle\langle\lambda_{\mathbf{p}}|. \quad (6.4)$$

The sum over λ is formal and includes integrals over continuous parameters like relative momenta.

Let's have a look at the full **two-point correlation function** $\langle\Omega| \text{T} \Phi(x)\Phi(y) |\Omega\rangle$ for scalar fields (the 'dressed propagator'). We start with $\langle\Omega| \Phi(x)\Phi(y) |\Omega\rangle$, whose analogue in the free theory is Eq. (2.72). Inserting the completeness relation, we can make the following observations:

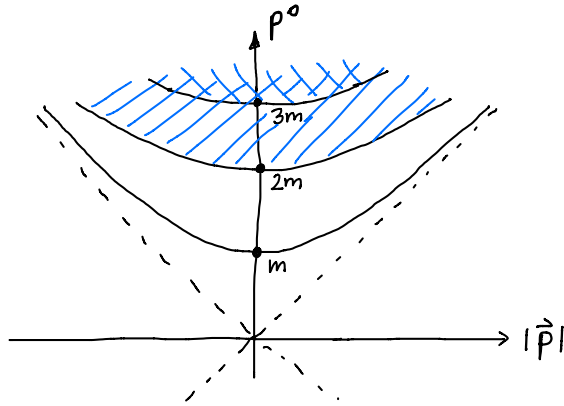


FIGURE 6.1: Eigenvalue spectrum of the Hamiltonian in terms of one-particle states with mass m and multiparticle states with invariant mass $m_\lambda \geq 2m$.

- Remembering Eq. (2.61), we infer that the field behaves under translations as $\Phi(x) = e^{ix \cdot P} \Phi(0) e^{-ix \cdot P}$. Since the vacuum is translationally invariant, the VEV of a single field $\langle \Omega | \Phi(x) | \Omega \rangle = \langle \Omega | \Phi(0) | \Omega \rangle$ must be a constant. We can always redefine the field by subtracting this constant so that the VEV vanishes. (For higher spin fields it vanishes automatically by Lorentz invariance.)
- For the matrix element $\langle \Omega | \Phi(x) | \lambda_{\mathbf{p}} \rangle$ we can also use translation invariance because $\lambda_{\mathbf{p}}$ is an eigenstate of P^μ :

$$\langle \Omega | \Phi(x) | \lambda_{\mathbf{p}} \rangle = \langle \Omega | e^{ix \cdot P} \Phi(0) e^{-ix \cdot P} | \lambda_{\mathbf{p}} \rangle = \langle \Omega | \Phi(0) | \lambda_{\mathbf{p}} \rangle e^{-ip \cdot x}. \quad (6.5)$$

If we denote by $U | \lambda_{\mathbf{0}} \rangle = | \lambda_{\mathbf{p}} \rangle$ a Lorentz boost from the rest frame to the momentum \mathbf{p} , we can further exploit Lorentz invariance from Eq. (2.65):

$$\langle \Omega | \Phi(0) | \lambda_{\mathbf{p}} \rangle = \langle \Omega | U \Phi(0) U^{-1} | \lambda_{\mathbf{p}} \rangle = \langle \Omega | \Phi(0) | \lambda_{\mathbf{0}} \rangle. \quad (6.6)$$

This quantity measures the overlap of $\langle \Omega | \Phi(0)$ with the state $| \lambda_{\mathbf{0}} \rangle$. For a one-particle state it is simply a constant, whereas for a general N -particle state it still depends on the relative momenta. In the following we will write

$$|\langle \Omega | \Phi(0) | \lambda_{\mathbf{0}} \rangle|^2 =: \frac{Z_{(\lambda)}}{(2\pi)^3}. \quad (6.7)$$

Compare this with the free theory, Eq. (2.35): If $\Phi(x)$ would only create a free particle from the vacuum, then the overlap would be $Z_{(1)} = 1$ for one-particle states and zero for all others. This is no longer true in an interacting theory because $\Phi(x)$ creates not only one-particle states. In the context of renormalization, we will later absorb $Z_{(1)}$ (which actually turns out to be infinite!) in the definition of the **renormalized field** so that the r.h.s. above becomes $1/(2\pi)^3$; however, this still leaves $Z_{(\lambda)} \neq 0$ for multiparticle states.

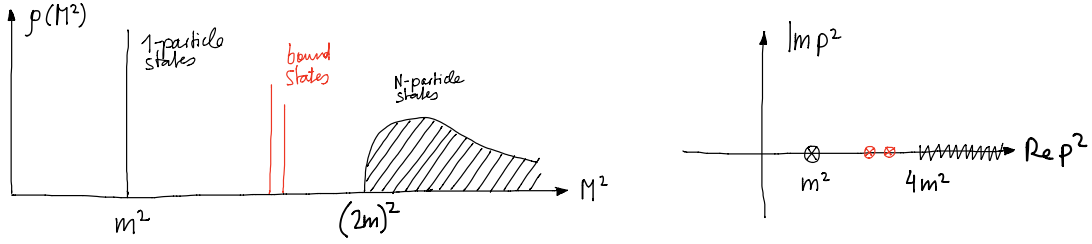


FIGURE 6.2: Left: spectral function of a typical field theory, with a single-particle peak at $M^2 = m^2$ and a multiparticle continuum for $M^2 \geq 4m^2$. Right: Analytic structure of the corresponding dressed propagator with single-particle (and potential bound-state) poles, together with a branch cut above $p^2 = 4m^2$.

For now, we arrive at

$$\langle \Omega | \Phi(x)\Phi(y) | \Omega \rangle = \frac{1}{(2\pi)^3} \sum_{\lambda} \int \frac{d^3p}{2E_p(\lambda)} e^{-ip(x-y)} Z_{(\lambda)}. \quad (6.8)$$

Including the time ordering, we can make the same manipulations for the integral as in the free theory, Eqs. (2.84) and (2.86), which leads to

$$\langle \Omega | \mathbb{T} \Phi(x)\Phi(y) | \Omega \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \sum_{\lambda} \frac{iZ_{(\lambda)}}{p^2 - m_{\lambda}^2 + i\epsilon} = \sum_{\lambda} Z_{(\lambda)} D_F(x-y, m_{\lambda}^2),$$

where we abbreviated the free Feynman propagator by

$$D_F(z, M^2) := \int \frac{d^4p}{(2\pi)^4} e^{-ipz} \frac{i}{p^2 - M^2 + i\epsilon}. \quad (6.9)$$

Remember that the sum over λ is a multi-dimensional integral over relative momenta. If we further define the **spectral function**

$$\rho(M^2) := \sum_{\lambda} 2\pi \delta(M^2 - m_{\lambda}^2) Z_{(\lambda)}, \quad (6.10)$$

then we arrive at the **Källén-Lehmann spectral representation**:

$$\langle \Omega | \mathbb{T} \Phi(x)\Phi(y) | \Omega \rangle = \int_0^{\infty} \frac{dM^2}{2\pi} \rho(M^2) D_F(x-y, M^2). \quad (6.11)$$

Therefore, the spectral function encodes the change from a free propagator to a dressed one. The spectral function for a typical theory is positive and has the form of Fig. 6.2. The one-particle states lead to an isolated δ -function peak at $M^2 = m^2$, which allows us to extract the squared mass m^2 of the particle as the **lowest-lying pole location** of the propagator in momentum space:

$$\int d^4x e^{ip(x-y)} \langle \Omega | \mathbb{T} \Phi(x)\Phi(y) | \Omega \rangle = \frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{4m^2}^{\infty} \frac{dM^2}{2\pi} \frac{i\rho(M^2)}{p^2 - M^2 + i\epsilon}, \quad (6.12)$$

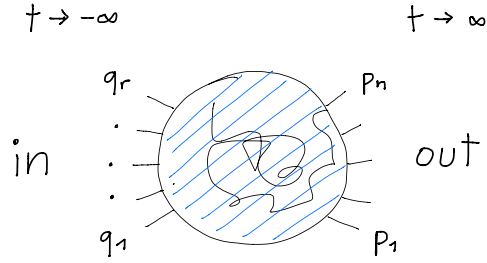


FIGURE 6.3: Idealized scattering process from r incoming to n outgoing particles.

where from now on we write $Z = Z_{(1)}$. The continuum of N -particle states begins at $M^2 \geq (2m)^2$, which leads to a branch cut in the propagator starting at $p^2 = 4m^2$. In addition, there could be further bound state poles below $M^2 = (2m)^2$. This property is usually more relevant in the context of composite fields (or higher n -point functions of elementary fields) but it can also happen in an elementary two-point function. For example, think of a scalar theory with a Φ^3 interaction: a particle can split into two, which contribute to the two-particle continuum, but in principle they could also form a scalar bound state with mass below $2m$.

S-matrix. The basic observables in scattering experiments are cross sections, which are related to the transition amplitudes that describe the scattering of incoming states $|g, \text{in}\rangle$ to outgoing states $|h, \text{out}\rangle$, cf. Fig. 6.3. Suppose that in the asymptotic past $t \rightarrow -\infty$ the state

$$|g, \text{in}\rangle = \int d^3 q_1 \cdots \int d^3 q_r g(\mathbf{q}_1, \dots, \mathbf{q}_r) a_{\text{in}}^\dagger(\mathbf{q}_1) \cdots a_{\text{in}}^\dagger(\mathbf{q}_r) |\Omega\rangle \quad (6.13)$$

describes a collection of wave packets (defined by the function g) that correspond to individual, well-separated single-particle states. When the particles approach each other, they start to interact and scatter into the final state $|h, \text{out}\rangle$, which for $t \rightarrow \infty$ describes again asymptotically free and well-separated 1-particle states:

$$|h, \text{out}\rangle = \int d^3 p_1 \cdots \int d^3 p_n h(\mathbf{p}_1, \dots, \mathbf{p}_n) a_{\text{out}}^\dagger(\mathbf{p}_1) \cdots a_{\text{out}}^\dagger(\mathbf{p}_n) |\Omega\rangle. \quad (6.14)$$

The in and out states are created from the interacting vacuum $|\Omega\rangle$ by action of the fields Φ_{in} and Φ_{out} at $t \rightarrow \pm\infty$. These are free fields that satisfy the free Klein-Gordon equation, however with mass $m \neq m_0$, which is the one-particle pole of the Feynman propagator of the full interacting theory, and energy $E_p = \sqrt{\mathbf{p}^2 + m^2}$. Therefore, we can expand Φ_{in} and Φ_{out} into Fourier modes with corresponding creation and annihilation operators.

lation operators:

$$\begin{aligned}\Phi_{\text{in}}(x) &= \int \frac{d^3p}{2E_p} (a_{\text{in}}(\mathbf{p}) f_p(x) + a_{\text{in}}^\dagger(\mathbf{p}) f_p^*(x)), \\ \Phi_{\text{out}}(x) &= \int \frac{d^3p}{2E_p} (a_{\text{out}}(\mathbf{p}) f_p(x) + a_{\text{out}}^\dagger(\mathbf{p}) f_p^*(x)), \\ f_p(x) &= \frac{1}{(2\pi)^{3/2}} e^{-ipx} \Big|_{p^0=E_p}.\end{aligned}\tag{6.15}$$

The question is: how is the full interacting field $\Phi(x)$ related to $\Phi_{\text{in}}(x)$ and $\Phi_{\text{out}}(x)$? What we will need in the following is that

$$\begin{aligned}\langle \alpha | \Phi(x) | \beta \rangle &\xrightarrow{t \rightarrow -\infty} C \langle \alpha | \Phi_{\text{in}}(x) | \beta \rangle, \\ \langle \alpha | \Phi(x) | \beta \rangle &\xrightarrow{t \rightarrow \infty} C \langle \alpha | \Phi_{\text{out}}(x) | \beta \rangle.\end{aligned}\tag{6.16}$$

This does not hold as an operator equation, i.e., the field $\Phi(x)$ does not simply become a free field for $t \rightarrow \pm\infty$. The corresponding statement is **Haag's theorem** which says, in short, that a field that is free at a given time remains free for all times. Since we cannot perform measurements with free fields, the corresponding quantum field theory would not have any empirical content. Hence, we only need Eq. (6.16) to hold in the weak sense, i.e., the matrix elements of $\Phi(x)$ should converge to those of $\Phi_{\text{in,out}}(x)$ in a suitable manner at $t \rightarrow \pm\infty$. For the overlap of $\Phi(x)$ between the vacuum and one-particle states this entails

$$\underbrace{|\langle \Omega | \Phi(0) | \lambda_0 \rangle|^2}_{Z/(2\pi)^3} = C^2 \underbrace{|\langle \Omega | \Phi_{\text{in}}(0) | \lambda_0 \rangle|^2}_{1/(2\pi)^3}\tag{6.17}$$

and therefore $C = \sqrt{Z}$, whereas the (momentum-dependent) overlap with multiparticle states $Z_{(\lambda)}$ must vanish for $t \rightarrow \pm\infty$. This can be intuitively understood as follows: although all interactions between the incoming and outgoing particles are switched off asymptotically, the self-interactions of the particles remain, which leads to $m \neq m_0$.

LSZ reduction formula. The operators $\Phi_{\text{in}}(x)$ and $\Phi_{\text{out}}(x)$ act on the same Hilbert space of a free theory. Hence, there must be an operator S (the **scattering operator**) that maps the out states onto the in states: $|g, \text{in}\rangle = S |g, \text{out}\rangle$. From this definition it follows that

$$S \text{ is unitary: } S^{-1} = S^\dagger, \quad S |\Omega\rangle = |\Omega\rangle, \quad \Phi_{\text{in}}(x) = S \Phi_{\text{out}}(x) S^{-1}.\tag{6.18}$$

The goal in the following will be to compute the transition amplitude or **S-matrix element**

$$\langle h, \text{out} | g, \text{in} \rangle = \langle h, \text{out} | S | g, \text{out} \rangle = \langle h, \text{in} | S | g, \text{in} \rangle.\tag{6.19}$$

For simplicity we will work directly with the matrix element

$$\langle p_1 \dots p_n, \text{out} | q_1 \dots q_r, \text{in} \rangle = \langle \Omega | a_{\text{out}}(\mathbf{p}_1) \dots a_{\text{out}}(\mathbf{p}_n) a_{\text{in}}^\dagger(\mathbf{q}_1) \dots a_{\text{in}}^\dagger(\mathbf{q}_r) | \Omega \rangle,\tag{6.20}$$

but keep in mind for the following discussion that we should really smear this with normalizable wave packets as in Eqs. (6.13–6.14).

The strategy in calculating the S-matrix element is to successively replace the creation and annihilation operators that appear in Eq. (6.20) by the fully interacting field $\Phi(x)$. To simplify the notation, we return to our definition of the Lorentz-invariant scalar product between fields in Eq. (1.22),

$$(\Psi, \Phi) := i \int d^3x \Psi^*(x) \overleftrightarrow{\partial}_0 \Phi(x) = i \int d^3x [\Psi^*(x) \dot{\Phi}(x) - \dot{\Psi}^*(x) \Phi(x)], \quad (6.21)$$

which is time-independent as long as $\Psi(x)$ and $\Phi(x)$ are solutions of the free Klein-Gordon equation. The relations

$$(f_p, f_{p'}) = 2E_p \delta^3(\mathbf{p} - \mathbf{p}'), \quad (f_p^*, f_{p'}^*) = -2E_p \delta^3(\mathbf{p} - \mathbf{p}'), \quad (f_p, f_{p'}^*) = 0 \quad (6.22)$$

then allow us to extract the Fourier coefficients of Eq. (6.15) as

$$\begin{aligned} a_{\text{in}}(\mathbf{p}) &= (f_p, \Phi_{\text{in}}), & a_{\text{out}}(\mathbf{p}) &= (f_p, \Phi_{\text{out}}), \\ a_{\text{in}}^\dagger(\mathbf{p}) &= -(f_p^*, \Phi_{\text{in}}), & a_{\text{out}}^\dagger(\mathbf{p}) &= -(f_p^*, \Phi_{\text{out}}). \end{aligned} \quad (6.23)$$

To begin with, we can write for any function $F(\mathbf{x})$:

$$\int_{-\infty}^{\infty} dt \frac{\partial}{\partial t} F(\mathbf{x}) = \lim_{t \rightarrow \infty} F(\mathbf{x}) - \lim_{t \rightarrow -\infty} F(\mathbf{x}). \quad (6.24)$$

Therefore, we can establish the relation

$$\begin{aligned} Z^{-1/2} \int_{-\infty}^{\infty} dt \partial_0 (f_p, \Phi) &= \lim_{t \rightarrow \infty} Z^{-1/2} (f_p, \Phi) - \lim_{t \rightarrow -\infty} Z^{-1/2} (f_p, \Phi) \\ &= (f_p, \Phi_{\text{out}}) - (f_p, \Phi_{\text{in}}) = a_{\text{out}}(\mathbf{p}) - a_{\text{in}}(\mathbf{p}), \end{aligned} \quad (6.25)$$

where we used Eq. (6.16). Remember that this only holds *inside* expectation values such as that in Eq. (6.20); it is not an operator identity because the identification of $Z^{-1/2} \Phi(x)$ with $\Phi_{\text{out}}(x)$, $\Phi_{\text{in}}(x)$ for $t \rightarrow \pm\infty$ is only valid in the weak sense. Note that the terms (f_p, Φ_{out}) and (f_p, Φ_{in}) in the second line are time-independent because $\Phi_{\text{out}}(x)$ and $\Phi_{\text{in}}(x)$ solve the Klein-Gordon equation, but (f_p, Φ) depends on time since $\Phi(x)$ is the interacting field. We can then work out its time derivative:

$$\begin{aligned} \partial_0 (f_p, \Phi) &= i \int d^3x \partial_0 [f_p^*(x) \overleftrightarrow{\partial}_0 \Phi(x)] \\ &= i \int d^3x [f_p^*(x) \partial_0^2 \Phi(x) - \partial_0^2 f_p^*(x) \Phi(x)], \end{aligned} \quad (6.26)$$

because the crossed terms cancel each other. The idea is now to shuffle the time derivative in the second term from f_p^* to Φ . Since f_p^* is a plane wave, cf. Eq. (6.15), we can convert the time derivative into a spatial derivative:

$$(\partial_0)^2 f_p^*(x) = -E_p^2 f_p^*(x) = -(\mathbf{p}^2 + m^2) f_p^*(x) = (\nabla^2 - m^2) f_p^*(x). \quad (6.27)$$

At this point we should remember that we will ultimately put this back into the S-matrix element (6.19) that is smeared with wave packets; otherwise the following partial integration cannot be justified because the surface terms would not vanish. In that case we obtain

$$\partial_0 (f_p, \Phi) = i \int d^3x f_p^*(x) (\partial_0^2 - \nabla^2 + m^2) \Phi(x) = i \int d^3x f_p^*(x) (\square + m^2) \Phi(x). \quad (6.28)$$

In total, Eq. (6.25) becomes

$$a_{\text{out}}(\mathbf{p}) = a_{\text{in}}(\mathbf{p}) + iZ^{-1/2} \int d^4x f_p^*(x) (\square + m^2) \Phi(x), \quad (6.29)$$

which again holds only inside the expectation value. Recall that $\Phi(x)$ does *not* satisfy the free Klein-Gordon equation, otherwise the integral would be zero.

Putting this back into the S-matrix element (6.20) and thereby replacing $a_{\text{out}}(\mathbf{p}_n)$, we can successively permute $a_{\text{in}}(\mathbf{p}_n)$ to the right until it annihilates on the vacuum. Each step generates a factor $2E_{p_n} \delta^3(\mathbf{p}_n - \mathbf{q}_j)$, together with another S-matrix element where two momenta are taken out. Therefore, they describe the scattering of $r - 1$ in states into $n - 1$ out states. From the perspective of the full S-matrix element they are **disconnected terms**, whereas the connected contribution comes from the second piece in Eq. (6.29):

$$iZ^{-1/2} \int d^4x f_p^*(x) (\square + m^2) \langle \Omega | a_{\text{out}}(\mathbf{p}_1) \dots \Phi(x) a_{\text{in}}^\dagger(\mathbf{q}_1) \dots a_{\text{in}}^\dagger(\mathbf{q}_r) | \Omega \rangle. \quad (6.30)$$

This completes the first step. Next, we want to repeat the procedure for $a_{\text{in}}^\dagger(\mathbf{q}_1)$ which appears to the right of $\Phi(x)$. However, in this case it is not sufficient to write $a_{\text{in}}^\dagger(\mathbf{q}_1) = a_{\text{out}}^\dagger(\mathbf{q}_1) + \dots$ because ultimately $a_{\text{out}}^\dagger(\mathbf{q}_1)$ should annihilate on the left, but we still need to interchange its position with $\Phi(x)$. Earlier we teased that it is the time-ordered propagator (with the Feynman prescription to integrate over poles), and time-ordered correlation functions, that will become important in the interacting theory. In fact, the next step is where the **time ordering** finally comes in:

$$\begin{aligned} \Phi(x) a_{\text{in}}^\dagger(\mathbf{q}) - a_{\text{out}}^\dagger(\mathbf{q}) \Phi(x) &= \\ &= (f_q^*, \Phi_{\text{out}}) \Phi(x) - \Phi(x) (f_q^*, \Phi_{\text{in}}) \\ &= i \int d^3y f_q(y) \overleftrightarrow{\frac{\partial}{\partial y^0}} \Phi_{\text{out}}(y) \Phi(x) - i \int d^3y f_q(y) \overleftrightarrow{\frac{\partial}{\partial y^0}} \Phi(x) \Phi_{\text{in}}(y) \\ &= iZ^{-1/2} \left[\lim_{y^0 \rightarrow \infty} \int d^3y f_q(y) \overleftrightarrow{\frac{\partial}{\partial y^0}} \Phi(y) \Phi(x) - \lim_{y^0 \rightarrow -\infty} \int d^3y f_q(y) \overleftrightarrow{\frac{\partial}{\partial y^0}} \Phi(x) \Phi(y) \right] \\ &= iZ^{-1/2} \int_{-\infty}^{\infty} dy^0 \frac{\partial}{\partial y^0} \int d^3y f_q(y) \overleftrightarrow{\frac{\partial}{\partial y^0}} \top \Phi(x) \Phi(y) \\ &= iZ^{-1/2} \int d^4y f_q(y) (\square_y + m^2) \top \Phi(x) \Phi(y). \end{aligned} \quad (6.31)$$

In the third equality we used the fact that the scalar products with Φ_{in} and Φ_{out} are time-independent, so we are free to shift the time variable $y^0 \rightarrow \pm\infty$ and replace the interacting field with the in and out fields (which holds inside matrix elements). In the fourth equality we used Eq. (6.24), and we finally repeated the steps that led us from Eq. (6.26) to (6.29). As desired, the second term $a_{\text{out}}^\dagger(\mathbf{q}) \Phi(x)$ on the l.h.s. will produce disconnected terms upon permuting it to the left, whereas the interacting part is generated by the r.h.s. of the equation.

In this way one can proceed until all creation and annihilation operators are replaced by the respective field operators. The final result is

$$\begin{aligned} \langle p_1 \dots p_n, \text{out} | q_1 \dots q_r, \text{in} \rangle_{\text{conn.}} &= (iZ^{-1/2})^{n+r} \left[\prod_{i=1}^n \int d^4 x_i f_{p_i}^*(x_i) (\square_{x_i} + m^2) \right] \times \\ &\times \left[\prod_{j=1}^r \int d^4 y_j f_{q_j}(y_j) (\square_{y_j} + m^2) \right] \langle \Omega | \mathbb{T} \Phi(x_1) \dots \Phi(x_n) \Phi(y_1) \dots \Phi(y_r) | \Omega \rangle. \end{aligned} \quad (6.32)$$

This is known as the **LSZ reduction formula** (Lehmann, Symanzik, Zimmermann). It reduces the computation of S-matrix elements to the calculation of the time-ordered **correlation functions** or simply **Green functions** of the fully interacting theory:

$$G(x_1 \dots x_n, y_1 \dots y_r) := \langle \Omega | \mathbb{T} \Phi(x_1) \dots \Phi(x_n) \Phi(y_1) \dots \Phi(y_r) | \Omega \rangle. \quad (6.33)$$

For further interpretation, we can use

$$\begin{aligned} \int d^4 x f_p^*(x) (\square + m^2) \Phi(x) &= \int d^4 x (\square + m^2) f_p^*(x) \Phi(x) \\ &= (-p^2 + m^2) \int d^4 x f_p^*(x) \Phi(x) \end{aligned} \quad (6.34)$$

for wave packets, which leads to the following form of the LSZ formula:

$$\begin{aligned} &\prod_{i=1}^n \int d^4 x_i f_{p_i}^*(x_i) \prod_{j=1}^r \int d^4 y_j f_{q_j}(y_j) G(x_1 \dots x_n, y_1 \dots y_r) \\ &= \left(\prod_{i=1}^n \frac{i\sqrt{Z}}{p_i^2 - m^2} \right) \left(\prod_{j=1}^r \frac{i\sqrt{Z}}{q_j^2 - m^2} \right) \langle p_1 \dots p_n, \text{out} | q_1 \dots q_r, \text{in} \rangle_{\text{conn.}}, \end{aligned} \quad (6.35)$$

plus further disconnected terms. The left-hand side is now just the Fourier transform of the Green function, i.e., the Green function in momentum space (modulo factors $(2\pi)^{3/2}$ from the f 's). Note that all momenta in the S-matrix element are onshell, $p_i^2 = q_j^2 = m^2$, because these are the physical momenta of 1-particle states. The prefactors on the r.h.s. are therefore singular; they correspond exactly to the pole contributions of the full propagator of the theory, cf. Eq. (6.12). Consequently, they must cancel with the l.h.s.: the Green function will contain a sum of terms with poles in the momenta, where only those terms survive in the connected S-matrix whose poles cancel exactly with the kinematic factors

$$\prod_{i=1}^n (p_i^2 - m^2) \prod_{j=1}^r (q_j^2 - m^2), \quad (6.36)$$

because all other contributions are not connected. Therefore, the recipe for calculating S-matrix elements is as follows:

- Calculate the Fourier transform of the Green function $G(x_1 \dots x_n, y_1 \dots y_r)$.
- Set all external momenta onshell: $p_i^2 = m^2$, $q_j^2 = m^2$. This generates a sum of terms that are distinguished by their pole structure.
- To obtain the connected S-matrix element, take the residue with respect to the $n + r$ pole factors.

This sounds straightforward enough, but the open question is: how can we actually calculate such Green functions?

7 Perturbation theory

The goal in the following is to calculate the n -point Green functions

$$\langle \Omega | \mathbb{T} \Phi(x_1) \dots \Phi(x_n) | \Omega \rangle \quad (7.1)$$

of a scalar theory in perturbation theory. That is, we assume that the interactions contained in $\mathcal{L}_{\text{int}} = \mathcal{L} - \mathcal{L}_0$ or $H_{\text{int}} = H - H_0$ are so weak that we can systematically expand these Green functions (and therefore also scattering amplitudes) in powers of the coupling constant. Perturbation theory has turned out to be immensely successful in many different contexts such as QED, the weak interaction, (to some extent) QCD, or also effective field theories.

Correlators in the interaction picture. How can we rearrange Eq. (7.1) in a form where \mathcal{L}_{int} appears explicitly? To begin with, recall Eq. (2.61) which follows from translation invariance and tells us how the field $\Phi(x)$ evolves in time:

$$\Phi(t, \mathbf{x}) = e^{iH(t-t_0)} \Phi(t_0, \mathbf{x}) e^{-iH(t-t_0)}. \quad (7.2)$$

$\Phi(x)$ is the field operator in the Heisenberg picture and carries the full spacetime dependence. Now let's define the **interaction picture field** $\Phi_I(x)$ as a field that 'evolves' with the Hamiltonian H_0 of the free theory:

$$\Phi_I(t, \mathbf{x}) = e^{iH_0(t-t_0)} \Phi_I(t_0, \mathbf{x}) e^{-iH_0(t-t_0)}. \quad (7.3)$$

By definition this is a free field that satisfies the free KG equation with mass m_0 , and we can expand it into Fourier modes as in Eq. (1.16). We now assume that the two fields are equal at some time t_0 , where they have the same functional form $\Phi(t_0, \mathbf{x}) = \Phi_I(t_0, \mathbf{x})$. In that case we can relate $\Phi(x)$ and $\Phi_I(x)$ at arbitrary time $x_0 = t$ by

$$\Phi(t, \mathbf{x}) = U^\dagger(t, t_0) \Phi_I(t, \mathbf{x}) U(t, t_0), \quad U(t, t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)}. \quad (7.4)$$

Note that $U(t, t_0) \neq e^{-iH_{\text{int}}(t-t_0)}$ because H does not commute with H_0 .

Actually the assumption $\Phi(t_0, \mathbf{x}) = \Phi_I(t_0, \mathbf{x})$ cannot hold in general, because Haag's theorem states that a free field will always remain free. That is, there is no unitary transformation that relates Φ to Φ_I , and consequently the interaction picture does not exist. We will ignore this problem in the following and hope that everything we do can still be justified in the sense of weakly converging matrix elements.

In any case, we can derive the following Schrödinger equation for the evolution operator $U(t, t_0)$:

$$\begin{aligned} i \frac{\partial U}{\partial t} &= e^{iH_0(t-t_0)} (H - H_0) e^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)} H_{\text{int}} e^{-iH_0(t-t_0)} U(t, t_0) =: H_I(t) U(t, t_0). \end{aligned} \quad (7.5)$$

$H_I(t)$ is the Hamiltonian in the interaction picture, i.e., H_{int} evolved with H_0 . It is simple because the functional dependence of $H_I(t)$ on $\Phi_I(t, \mathbf{x})$ is the same as that of H_{int} on $\Phi(t_0, \mathbf{x})$, for example in Φ^4 theory:

$$H_{\text{int}}(t_0) = \int d^3x \frac{\lambda}{4!} \Phi(t_0, \mathbf{x})^4 \quad \Rightarrow \quad H_I(t) = \int d^3x \frac{\lambda}{4!} \Phi_I(t, \mathbf{x})^4. \quad (7.6)$$

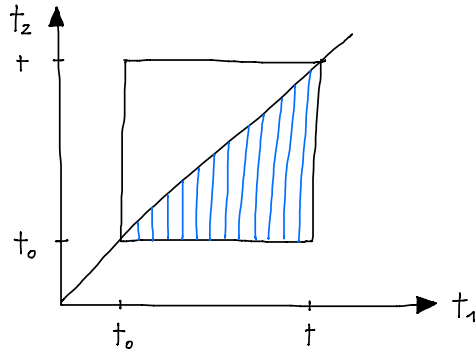


FIGURE 7.1: Symmetric integration domain in Eq. (7.7).

Therefore, the solution of (7.5) allows us to express the full field $\Phi(t, \mathbf{x})$ in terms of the interaction-picture Hamiltonian $H_I(t)$ and ultimately the interaction-picture field $\Phi_I(t, \mathbf{x})$, which is simple to handle because it is a free field that can be expanded into Fourier modes.

Another remark is in order: the explicit form for $U(t, t_0)$ in Eq. (7.4) only holds for the case where H , H_0 and H_{int} are all time-independent. This is true for the full Hamiltonian H but in general not for $H_0(t)$ and $H_{\text{int}}(t)$: $\dot{H}_{\text{int}}(t) = i[H, H_{\text{int}}(t)] = i[H_0(t), H_{\text{int}}(t)]$. Fortunately, it is not necessary to specify $U(t, t_0)$ explicitly: one can show that the generic relation between $\Phi(x)$ and $\Phi_I(x)$ in Eq. (7.4) leads to the same Schrödinger equation.

Eq. (7.5) is solved by

$$U(t, t_0) = 1 + (-i) \int_{t_0}^t dt_1 H_I(t_1) + (-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) + \dots \quad (7.7)$$

To see this, take the time derivative with $(\partial/\partial t) \int_{t_0}^t dt' f(t') = f(t)$: each term in the series reproduces the previous one with a factor $-iH_I(t)$, and the initial condition $U(t_0, t_0) = 1$ is satisfied. Note that the factors H_I in the integrand are automatically time-ordered because $t_1 > t_2 > t_3 > \dots$, so we can equally write

$$U(t, t_0) = 1 + (-i) \int_{t_0}^t dt_1 H_I(t_1) + \frac{(-i)^2}{2!} \int_{t_0}^t dt_1 \int_{t_0}^t dt_2 \mathbb{T} \{H_I(t_1) H_I(t_2)\} + \dots \quad (7.8)$$

Here we additionally exploited the fact that the integral is symmetric in t_1 and t_2 , cf. Fig. 7.1; this also holds for the higher-order diagrams. The series defines the **time-ordered exponential**

$$U(t, t_0) =: \mathbb{T} \exp \left[-i \int_{t_0}^t dt' H_I(t') \right] \quad (7.9)$$

as the time ordering of the individual terms in the series expansion. From the expansion one can also prove the properties

$$\begin{aligned} U^\dagger(t_1, t_2) &= U^{-1}(t_1, t_2) = U(t_2, t_1), \\ U(t_1, t_2) U(t_2, t_3) &= U(t_1, t_3) \quad \text{for } t_1 \geq t_2 \geq t_3. \end{aligned} \quad (7.10)$$

The hermitian conjugation switches all i factors and can be reversed by exchanging the integration limits, which leads to the first relation. To verify the second one, observe that $U(t, t_2) U(t_2, t_3)$ satisfies the same Schrödinger equation (7.5) so it can be written as $U(t, t')$, and the boundary condition $U(t_3, t') = 1$ entails $t' = t_3$.

With Eqs. (7.4) and (7.10) at hand, we can work out the full two-point function. Consider first the case $x^0 > y^0$:

$$\begin{aligned} \langle \Omega | \Phi(x) \Phi(y) | \Omega \rangle &= \langle \Omega | U^\dagger(x_0, t_0) \Phi_I(x) U(x_0, t_0) U^\dagger(y_0, t_0) \Phi_I(y) U(y_0, t_0) | \Omega \rangle \\ &= \langle \Omega | U(t_0, x_0) \Phi_I(x) U(x_0, y_0) \Phi_I(y) U(y_0, t_0) | \Omega \rangle. \end{aligned} \quad (7.11)$$

Let's insert some large time $T \gg x^0, y^0$:

$$\dots = \langle \Omega | U(t_0, T) \underbrace{U(T, x_0) \Phi_I(x) U(x_0, y_0) \Phi_I(y) U(y_0, -T)}_{\text{time-ordered}} U(-T, t_0) | \Omega \rangle. \quad (7.12)$$

The bracket is then already time-ordered, so we can put a time-ordering symbol in front of it and combine all U 's inside:

$$\dots = \langle \Omega | U(t_0, T) \mathbb{T} \left\{ \Phi_I(x) \Phi_I(y) U(T, -T) \right\} U(-T, t_0) | \Omega \rangle. \quad (7.13)$$

Since this is time-ordered, the opposite case with $y^0 > x^0$ gives the same result, and therefore the full correlator becomes

$$\langle \Omega | \mathbb{T} \Phi(x) \Phi(y) | \Omega \rangle = \langle \Omega | U(t_0, T) \mathbb{T} \left\{ \Phi_I(x) \Phi_I(y) U(T, -T) \right\} U(-T, t_0) | \Omega \rangle. \quad (7.14)$$

The quantity $U(T, -T)$ is given by

$$U(T, -T) = \mathbb{T} \exp \left[-i \int_{-T}^T dt H_I(t) \right] = \mathbb{T} e^{iS_I}, \quad (7.15)$$

where $S_I = \int d^4x \mathcal{L}_I = - \int d^4x \mathcal{H}_I$ is the action corresponding to the interacting part that depends on the field $\Phi_I(x)$. We assumed that the interacting Lagrangian contains no field derivatives so that $\mathcal{L}_I = -\mathcal{H}_I$. The expression (7.14) still depends on the arbitrary reference time t_0 and the interacting vacuum $|\Omega\rangle$ which we have to get rid of.

Free vs. interacting vacuum. We would like to relate the full interacting vacuum $|\Omega\rangle$ to the vacuum $|0\rangle$ of the free theory. To do so, recall that $H|\Omega\rangle = 0$ and $\langle \Omega | \Omega \rangle = 1$. That is, we ‘renormalized’ the interacting theory so that the vacuum energy $E_\Omega = 0$, which we motivated with the arbitrary counterterm V_0 in the Lagrangian. However, doing so removes our freedom to set the vacuum energy in the corresponding free theory

(defined by H_0) to zero: $H_0 |n\rangle = E_n |n\rangle$ with $E_0 \leq E_1 \leq E_2 \leq \dots$, but $E_0 \neq 0$. In any case we can write

$$U(-T, t_0) |\Omega\rangle = e^{-iH_0(T+t_0)} \underbrace{e^{iH(T+t_0)} |\Omega\rangle}_{=|\Omega\rangle} = \sum_{n=0}^{\infty} e^{-iE_n(T+t_0)} |n\rangle \langle n|\Omega\rangle.$$

In the last step we inserted a complete set of states of the free theory. Assuming that $\langle 0|\Omega\rangle \neq 0$, we can eliminate the contributions from the states with higher energy by taking the limit $T \rightarrow \infty(1-i\epsilon)$ because this will eliminate all contributions from the energies $E_n > E_0$:

$$U(-T, t_0) |\Omega\rangle \xrightarrow{T \rightarrow \infty(1-i\epsilon)} e^{-iE_0(T+t_0)} |0\rangle \langle 0|\Omega\rangle =: c(-T, t_0) |0\rangle.$$

The analogous case for $\langle \Omega|$ gives

$$\langle \Omega| U^\dagger(T, t_0) \xrightarrow{T \rightarrow \infty(1-i\epsilon)} \langle 0| c^*(T, t_0). \quad (7.16)$$

After plugging this into Eq. (7.14) we are still left with the awkward factors $c(-T, t_0)$ and $c^*(T, t_0)$. We can remove them too by noting that

$$\begin{aligned} \langle \Omega|\Omega\rangle &= 1 \xrightarrow{T \rightarrow \infty(1-i\epsilon)} c^*(T, t_0) c(-T, t_0) \langle 0| U(T, t_0) U^\dagger(-T, t_0) |0\rangle \\ &= c^*(T, t_0) c(-T, t_0) \langle 0| U(T, -T) |0\rangle. \end{aligned} \quad (7.17)$$

Inserting everything into Eq. (7.14) we arrive at the final result for the full propagator:

$$\langle \Omega| \mathbb{T} \Phi(x) \Phi(y) |\Omega\rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0| \mathbb{T} \{ \Phi_I(x) \Phi_I(y) U(T, -T) \} |0\rangle}{\langle 0| U(T, -T) |0\rangle}. \quad (7.18)$$

It can be generalized to arbitrary n -point functions:

$$\langle \Omega| \mathbb{T} \Phi(x_1) \dots \Phi(x_n) |\Omega\rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0| \mathbb{T} \{ \Phi_I(x_1) \dots \Phi_I(x_n) e^{iS_I} \} |0\rangle}{\langle 0| \mathbb{T} e^{iS_I} |0\rangle}. \quad (7.19)$$

With this formula we have in principle everything in place to do perturbation theory. We could expand e^{iS_I} in the small coupling constant, express Φ_I in terms of creation and annihilation operators (since it is a free field), take the time ordering, and calculate any correlation function simply by brute force. However, this also becomes quite repetitive and cumbersome, which is where Wick's theorem comes to rescue.

Wick's theorem. To shorten the notation, we will write the interaction-picture field as $\Phi_I(x) = \phi(x)$. Since it is a free field, we can decompose it into positive- and negative-frequency parts:

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} (a_p e^{-ipx} + a_p^\dagger e^{ipx}) = \phi_+(x) + \phi_-(x), \quad (7.20)$$

with $\phi_+(x) |0\rangle = 0 = \langle 0| \phi_-(x)$. In the following we want to express products of field operators in terms of their normal ordered versions, which means that all creation operators are shuffled to the left and all annihilation operators to the right or, equivalently, all instances of $\phi_-(x)$ go to the left and all instances of $\phi_+(x)$ to the right.

Consider the product of two fields $\phi(x)\phi(y)$. In terms of positive- and negative-frequency modes it has the form

$$\begin{aligned}\phi(x)\phi(y) &= \phi_+(x)\phi_+(y) + \phi_+(x)\phi_-(y) + \phi_-(x)\phi_+(y) + \phi_-(x)\phi_-(y) \\ &= :\phi(x)\phi(y): + [\phi_+(x), \phi_-(y)],\end{aligned}\quad (7.21)$$

and likewise $\phi(y)\phi(x) = :\phi(x)\phi(y): + [\phi_+(y), \phi_-(x)]$. Inserting the Fourier modes, we find for the commutator

$$[\phi_+(x), \phi_-(y)] = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_p} e^{-ip(x-y)} \Big|_{p^0=E_p} = D(x-y), \quad (7.22)$$

where $D(x-y)$ has been given in Eq. (2.72). In total this yields

$$\begin{aligned}\mathbb{T}\phi(x)\phi(y) &= :\phi(x)\phi(y): + \Theta(x^0 - y^0)D(x-y) + \Theta(y^0 - x^0)D(y-x) \\ &= :\phi(x)\phi(y): + D_F(x-y).\end{aligned}\quad (7.23)$$

Since $\langle 0|\mathcal{O}|0\rangle = 0$, this implies for the vacuum expectation value

$$\langle 0|\mathbb{T}\phi(x)\phi(y)|0\rangle = D_F(x-y), \quad (7.24)$$

which is just our earlier definition of the Feynman propagator (remember that $\phi(x)$ is a free field and $|0\rangle$ the free vacuum).

What is useful about the identity is that we can immediately generalize it to arbitrary n -point functions. This is known as **Wick's theorem**, and it states that the time-ordered product $\mathbb{T}\phi(x_1)\dots\phi(x_n)$ is equal to the normal-ordered product $:\phi(x_1)\dots\phi(x_n):$ plus all possible combinations of normal orderings and *contractions* of distinct fields. A contraction of two fields $\phi(x_1), \phi(x_2)$ is defined to be equal to the Feynman propagator $D(x_1 - x_2)$ and denoted by

$$\overline{\phi(x)\phi(y)} = D_F(x-y). \quad (7.25)$$

Using the shorthand notation $\phi(x_i) = \phi_i$ and $D_F(x_i - x_j) = D_{ij}$, let's illustrate the result for the four-point function:

$$\begin{aligned}\mathbb{T}\{\phi_1\phi_2\phi_3\phi_4\} &= :\phi_1\phi_2\phi_3\phi_4: \\ &\quad + D_{12}:\phi_3\phi_4: + D_{13}:\phi_2\phi_4: + D_{14}:\phi_2\phi_3: \\ &\quad + D_{23}:\phi_1\phi_4: + D_{24}:\phi_1\phi_3: + D_{34}:\phi_1\phi_2: \\ &\quad + D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}.\end{aligned}\quad (7.26)$$

The Wick theorem for arbitrary n -point functions can be proven via induction (see Peskin-Schroeder, p.90). Only the last line above survives when taking vacuum expectation values, and therefore the VEV of a time-ordered product of fields equals the sum over all possible contractions:

$$\langle 0|\mathbb{T}\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle = D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}. \quad (7.27)$$

If n is odd, the VEV vanishes because there is always an odd number of normal-ordered fields remaining.

Feynman diagrams. A diagrammatic way to visualize such contractions is to draw Feynman diagrams: draw a point for each spacetime argument x_i and connect them by lines, which represent the Feynman propagators of the free theory. The four-point function from Eq. (7.27) then becomes

$$\langle 0 | \mathbb{T} \{ \phi_1 \phi_2 \phi_3 \phi_4 \} | 0 \rangle = \begin{array}{c} 1 \text{---} 2 \\ \text{---} \\ 3 \text{---} 4 \end{array} + \begin{array}{c} | \\ | \end{array} + \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}. \quad (7.28)$$

More interesting are expressions that contain more than one field at the same spacetime point, which leads to **loop diagrams**. Let's put Eq. (7.18) for the two-point function in ϕ^4 theory to use. When we expand the exponential in the numerator to $\mathcal{O}(\lambda)$ we obtain

$$\begin{aligned} \langle 0 | \mathbb{T} \phi(x) \phi(y) e^{-i \frac{\lambda}{4!} \int d^4 z \phi(z)^4} | 0 \rangle &= \\ &= \langle 0 | \mathbb{T} \phi(x) \phi(y) | 0 \rangle - i \frac{\lambda}{4!} \langle 0 | \mathbb{T} \phi(x) \phi(y) \int d^4 z \phi(z)^4 | 0 \rangle + \dots \end{aligned} \quad (7.29)$$

The first term is just the propagator line from x to y . Applying the Wick theorem to the combination $\phi(x)\phi(y)\phi(z)^4$ yields only two distinct expressions:

- If we contract $\phi(x)$ with $\phi(y)$, there are three distinguishable ways how to contract $\phi(z)$ with $\phi(z)$:

$$\overbrace{\phi(x) \phi(y)} \overbrace{\phi(z) \phi(z)} \overbrace{\phi(z) \phi(z)}.$$

- if we contract $\phi(x)$ with $\phi(z)$ (four possibilities) and $\phi(y)$ with $\phi(z)$ (three possibilities), there is one possibility left how to contract $\phi(z)$ with $\phi(z)$:

$$\overbrace{\phi(x) \phi(y) \phi(z) \phi(z)} \overbrace{\phi(z) \phi(z)}.$$

In total, this gives

$$\langle 0 | \mathbb{T} \{ \phi_x \phi_y \phi_z \phi_z \phi_z \phi_z \} | 0 \rangle = 3 \cdot D_{xy} D_{zz} D_{zz} + 4 \cdot 3 \cdot D_{xz} D_{yz} D_{zz}, \quad (7.30)$$

or in terms of diagrams:

$$\langle 0 | \mathbb{T} \left\{ \phi_x \phi_y \int_z \phi_z \phi_z \phi_z \phi_z \right\} | 0 \rangle = 3 \left(\begin{array}{c} \bullet \text{---} \bullet \\ x \quad y \end{array} \begin{array}{c} \circlearrowleft \\ z \end{array} \right) + 4 \cdot 3 \left(\begin{array}{c} \bullet \text{---} \bullet \\ x \quad z \quad y \end{array} \begin{array}{c} \circlearrowleft \\ z \end{array} \right), \quad (7.31)$$

where we abbreviated $\int d^4 z = \int_z$.

Clearly, for higher products of fields the number of possible Wick contractions will rise dramatically. Fortunately, however, this number *almost* cancels with the factors $4!$ from the denominators in the Taylor expansion. In the example above, the final prefactors are $1/8$ and $1/2$. Their denominators 8 and 2 are called **symmetry factors** of the diagrams, because they count the number of possibilities to exchange the components without changing the diagram itself. For example:

$$\begin{array}{c} \circlearrowleft \\ \bullet \end{array} \rightarrow 2 \times 2 \times 2 \quad \begin{array}{c} \circlearrowleft \\ \bullet \end{array} \rightarrow 2 \quad \begin{array}{c} \circlearrowleft \\ \bullet \end{array} \rightarrow 3! \quad \begin{array}{c} \circlearrowleft \\ \bullet \end{array} \rightarrow 3! \cdot 2 \quad (7.32)$$

In the first diagram we can flip both the upper and the lower loop horizontally, and we can exchange the loops vertically, which gives $2 \times 2 \times 2 = 8$. In the second diagram we can only do a horizontal flip, so the symmetry factor is 2. In the third diagram there are $3! = 6$ possibilities to exchange the three internal lines, and in the fourth diagram we can additionally perform a horizontal flip ($3! \times 2 = 12$). Note that the external points x and y are fixed and cannot be flipped.

Feynman rules. These observations hold in general and can be summarized by the Feynman rules. Consider an n -point function for a theory with a ϕ^m interaction:

$$\langle 0 | T \phi(x) \dots \phi(x_n) e^{-i \frac{\lambda}{m!} \int d^4 z \phi(z)^m} | 0 \rangle. \tag{7.33}$$

You can find all diagrams at a given order $\mathcal{O}(\lambda^k)$ in perturbation theory if you draw

- n external points x_i ,
 - k internal points z_j (**vertices**) with m incoming lines,
 - connect all points by Feynman propagators of the free theory,
 - divide each diagram by its symmetry factor,
 - and sum up all diagrams in the end.
- $x \bullet \text{---} \bullet = 1$
 $\begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} = (-i\lambda) \int d^4 z$
 $x \bullet \text{---} \bullet \xleftarrow{p} y = D_F(x-y)$

It is usually more convenient to write the Feynman propagator in momentum space:

$$D_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-y)} D_F(p), \quad D_F(p) = \frac{i}{p^2 - m_0^2 + i\epsilon}. \tag{7.34}$$

We will use the convention that the momentum points from right to left, i.e., from y to x . This is irrelevant for a scalar propagator because $D_F(x-y) = D_F(y-x)$, but the distinction will become important when we generalize the Feynman rules to fermions.


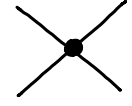

As an example, let's work out the 'tadpole' diagram, now with the abbreviation $\int_p = \int \frac{d^4 p}{(2\pi)^4}$:

$$= \frac{1}{2} (-i\lambda) \int d^4 z D_F(x-z) D_F(y-z) D_F(z-z)$$

$$= -\frac{i\lambda}{2} \int d^4 z \int \int \int_p \int_q \int_k e^{-ip(x-z)} e^{iq(y-z)} D_F(p) D_F(q) D_F(k) \tag{7.35}$$

$$= -\frac{i\lambda}{2} \int \int \int_p \int_q \int_k e^{-ipx} e^{iqy} D_F(p) D_F(q) D_F(k) (2\pi)^4 \delta^4(p-q).$$


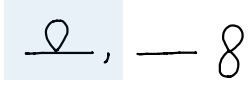
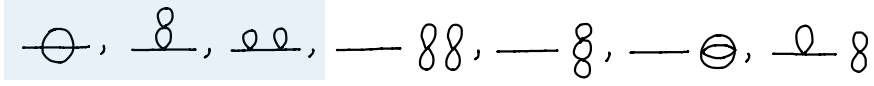
From here one can read off the **Feynman rules ‘in momentum space’** (this is a bit of a misnomer because the Green function is still given in real space), which are easier to handle in practice:

-)  = e^{-ipx}
-)  = $-i\lambda(2\pi)^4 \delta^4(\sum p_i - \sum q_j)$
-)  = $D_F(p) = \frac{i}{p^2 - m_0^2 + i\varepsilon}$
-) integrate over all momenta: $\int \frac{d^4p}{(2\pi)^4}$
-) divide by the symmetry factor.

Propagator in ϕ^4 theory. Let's put the Feynman rules to use and calculate the propagator of ϕ^4 theory, i.e., all diagrams that contribute to

$$\langle 0 | \text{T} \phi(x) \phi(y) e^{-i \frac{\lambda}{m!} \int d^4z \phi(z)^m} | 0 \rangle \quad (7.36)$$

up to $\mathcal{O}(\lambda^k)$. The propagator has two external points x and y , and a diagram at $\mathcal{O}(\lambda^k)$ has k vertices. Here is the complete list up to $\mathcal{O}(\lambda^2)$:

- $\mathcal{O}(\lambda^0)$: 
- $\mathcal{O}(\lambda^1)$: 
- $\mathcal{O}(\lambda^2)$: 

Observe that we arrive at the same result if we multiply the sum of all **connected diagrams** (those in the boxes) by the sum of all **vacuum bubbles**:

$$\left[\text{---} + \text{---} \text{---} + \text{---} \text{---} + \text{---} \text{---} + \text{---} \text{---} + \dots \right] \times \left[1 + \text{---} + \text{---} \text{---} + \text{---} \text{---} + \text{---} \text{---} + \dots \right].$$

This is not a coincidence because so far we have only dealt with the numerator in Eq. (7.19). To obtain the full Green function, we should also take into account the denominator

$$\langle 0 | \text{T} e^{-i \frac{\lambda}{m!} \int d^4z \phi(z)^m} | 0 \rangle, \quad (7.37)$$

which is called the **partition function**. Its perturbative expansion generates just these vacuum bubbles:

$$\mathcal{O}(\lambda^0) : 1 \quad \mathcal{O}(\lambda^1) : \text{loop} \quad \mathcal{O}(\lambda^2) : \text{two loops}, \text{bubble}, \text{circle}$$

so they factor out in the full Green function. Therefore we find


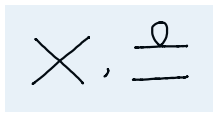
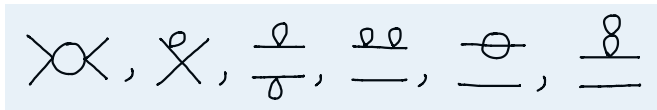
$$\langle \Omega | \mathbb{T} \Phi(x) \Phi(y) | \Omega \rangle = \sum (\text{partially connected terms}). \tag{7.38}$$

The meaning of ‘partially connected’ will become clear in a moment.

Four-point function in ϕ^4 theory. As another example, let’s have a look at the four-point function

$$\frac{\langle 0 | \mathbb{T} \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) e^{-i \frac{\lambda}{m!} \int d^4z \phi(z)^m} | 0 \rangle}{\langle 0 | \mathbb{T} e^{-i \frac{\lambda}{m!} \int d^4z \phi(z)^m} | 0 \rangle}. \tag{7.39}$$

It has four external points and k vertices at $\mathcal{O}(\lambda^k)$. Ignoring pure vacuum bubbles, the diagrams up to $\mathcal{O}(\lambda^2)$ are given by

- $\mathcal{O}(\lambda^0)$: 
- $\mathcal{O}(\lambda^1)$: 
- $\mathcal{O}(\lambda^2)$: 

We already found the zeroth-order result in Eq. (7.28) as the sum of the three disconnected terms. Since they all have the same structure we have represented them here by a single diagram for brevity. The same goes for the other diagrams where we have only drawn one representative for each case, e.g. for the fourth diagram at $\mathcal{O}(\lambda^2)$: we can attach the two bubbles at the upper and lower line, and there are three permutations of the two lines. Here it also becomes clear why we referred to ‘partially connected’ terms in Eq. (7.38): the full Green function is the sum of those diagrams where we can no longer factor out vacuum bubbles, but they do not need to be fully connected.

1-particle irreducible diagrams. A class of diagrams that are important for theoretical analyses are the 1PI (one-particle irreducible) diagrams. The 1PI property is defined as follows: consider only diagrams which are fully connected. Remove (‘amputate’) its external legs. If the diagram is still connected after cutting a single internal line, it is 1PI. Some examples and counterexamples are:



Up to $\mathcal{O}(\lambda^2)$, the 1PI contributions to the propagator and the four-point function are therefore the following:

$$\text{---}\text{---}\text{---} + \text{---}\text{---}\text{---} + \text{---}\text{---}\text{---} \quad \text{and} \quad \text{---}\text{---} + \text{---}\text{---}\text{---}.$$

Now let's denote the full propagator by

$$G(x-y) = \langle \Omega | \mathsf{T} \Phi(x) \Phi(y) | \Omega \rangle = \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-y)} G(p) = \text{---}\bullet\text{---} \quad (7.40)$$

and define the **self-energy** of the scalar particle as the sum of all 1PI graphs for the 2-point function:

$$\frac{\Sigma(p)}{i} := \text{---}\text{---}\text{---} = \text{---}\text{---}\text{---} + \text{---}\text{---}\text{---} + \text{---}\text{---}\text{---} + \dots$$

Observe that we can obtain the propagator by resumming its 1PI contributions:

$$\begin{aligned} G(p) &= \text{---}\bullet\text{---} = \text{---}\text{---}\text{---} + \text{---}\text{---}\text{---} + \text{---}\text{---}\text{---} + \dots \\ &= D_F \left[1 + \frac{\Sigma}{i} \left(D_F + D_F \frac{\Sigma}{i} D_F + \dots \right) \right] \end{aligned} \quad (7.41)$$

$$= D_F \left[1 + \frac{\Sigma}{i} G(p) \right], \quad (7.42)$$

and therefore

$$iG^{-1}(p) = p^2 - m_0^2 - \Sigma(p) \quad \Leftrightarrow \quad G(p) = \frac{i}{p^2 - m_0^2 - \Sigma(p) + i\epsilon}. \quad (7.43)$$

On the other hand, we know from the Källén-Lehmann spectral representation (6.12) that the full propagator must have the form

$$G(p) = \frac{iZ}{p^2 - m^2 + i\epsilon} + \text{terms that are regular at } p^2 = m^2. \quad (7.44)$$

In this sense the self-interactions of the particle (the quantum loop corrections) shift its mass from m_0 to m , so that the pole appears at $p^2 = m^2$, and $\Sigma(p)$ takes indeed the meaning of a self-energy.

What we have done here is resumming the geometric series. For illustration, replace $\Sigma \rightarrow x$, $D_F \rightarrow i$ and $G \rightarrow if(x)$:

$$f(x) = 1 + x + x^2 + \dots = 1 + x(1 + x + \dots) = 1 + xf(x) \quad \Rightarrow \quad f(x) = \frac{1}{1-x}. \quad (7.45)$$

Of course this is only justified for $|x| < 1$, i.e., as long as the coupling is small. Fortunately, Eq. (7.43) can be also derived nonperturbatively: it is the Dyson-Schwinger equation for the propagator, which is an exact equation:

$$\text{---}\bullet\text{---}^{-1} = \text{---}\text{---}\text{---}^{-1} + \text{---}\text{---}\text{---} + \text{---}\text{---}\text{---}. \quad (7.46)$$

All ingredients with filled blobs are dressed. In our example this means that

$$f(x) = \frac{1}{1-x} = 1 + xf(x) = 1 + x + x^2 f(x) = \dots \quad (7.47)$$

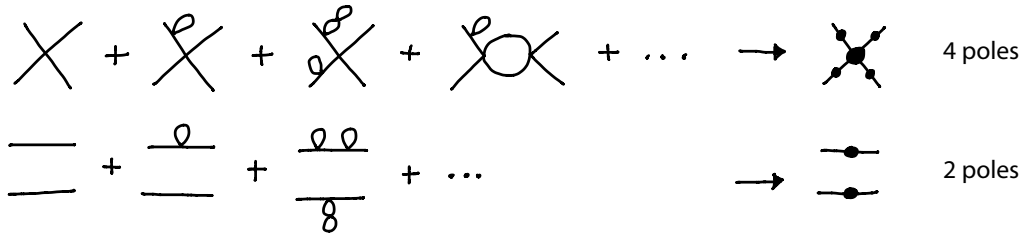
is valid for all x except $x = 1$, because there is always a remainder that reproduces the exact result, whereas the geometric series $f(x) = \sum_{n=0}^{\infty} x^n$ converges to the exact value only for $|x| < 1$. Hence, the Dyson-Schwinger equation is more general than the perturbative expansion.

There may be also genuinely nonperturbative effects that are not reproducible by the perturbative series, not even for a small coupling. An example is QCD in the chiral limit, where the dressed propagator has a nonvanishing mass function even if the mass in the Lagrangian is zero. This effect is due to spontaneous chiral symmetry breaking; although it follows from the Dyson-Schwinger equation, it cannot be achieved at any order in perturbation theory.

In a similar way one can generate higher n -point functions from their 1PI counterparts because they can only differ by internal (fully resummed) propagator lines. Hence, the 1PI correlation functions encode the ‘irreducible’ content of an n -point interaction. In Sec. 8 we will also see that they are convenient for discussing the renormalization of the quantum field theory.

Scattering amplitude. With all that in mind, we can now go back to the scattering amplitude and the LSZ formula (6.35). There we found that the full Green function is proportional to the connected S-matrix element, with one pole of mass m attached for each external particle, plus further disconnected diagrams. We argued that the Green function will be a sum of terms with different pole factors, and only those terms survive in the S-matrix element where the number of poles matches exactly.

From the discussion above it is clear that such pole factors can only come from fully resummed propagators with mass m . This means that only **connected** terms in the Green function can contribute to the S-matrix, for example:



Since the external particles are onshell, removing the pole factors is equivalent to removing the dressed propagators according to Eq. (7.44). We ignore the remaining Z factors because in the process of renormalization we will absorb them into the fields. In that way we arrive at the final result for the S-matrix element expressed through the renormalized field $\Phi(x)$, which we write in terms of the **invariant amplitude** \mathcal{M} :

$$\begin{aligned} (2\pi)^4 \delta^4\left(\sum p_i - \sum q_j\right) i\mathcal{M} &:= (2\pi)^{\frac{3}{2}(n+r)} \langle p_1 \dots p_n, \text{out} | q_1 \dots q_r, \text{in} \rangle_{\text{conn.}} \\ &= \text{FT} \langle \Omega | \text{T} \Phi(x_1) \dots \Phi(y_r) | \Omega \rangle_{\substack{\text{connected,} \\ \text{amputated,} \\ \text{onshell}}} \end{aligned} \quad (7.48)$$

Since all external momenta are onshell, they describe physical particles with $p_i^2 = m^2$. The internal propagators, whose loop momenta are integrated over, are offshell and correspond to **virtual particles** with $k^2 \neq m^2$ (note that four-momentum conservation

is still satisfied at each vertex). In this sense the scattering amplitude is the summation over all possible virtual processes that can contribute.

What remains to be done is to state the Feynman rules for the scattering matrix element. Let's derive them explicitly for the 1-loop graph at $\mathcal{O}(\lambda^2)$:

$$= I(x_1, x_2, y_1, y_2) \quad (7.49)$$

Employing the Feynman rules in momentum space, the diagram takes the form

$$\begin{aligned} \dots &= \frac{(-i\lambda)^2}{2} \int_{p_1} \int_{p_2} \int_{q_1} \int_{q_2} \int_{k_1} \int_{k_2} (2\pi)^4 \delta^4(p_1 + p_2 - k_1 - k_2) (2\pi)^4 \delta^4(k_1 + k_2 - q_1 - q_2) \\ &\quad \times e^{-ip_1 x_1} e^{-ip_2 x_2} e^{iq_1 y_1} e^{iq_2 y_2} \\ &\quad \times D_F(p_1) D_F(p_2) D_F(q_1) D_F(q_2) D_F(k_1) D_F(k_2). \end{aligned} \quad (7.50)$$

For the S-matrix element we need the amplitude in momentum space, so we take the Fourier transform

$$\begin{aligned} &\int d^4 x_1 e^{ip_1 x_1} \int d^4 x_2 e^{ip_2 x_2} \int d^4 y_1 e^{-iq_1 y_1} \int d^4 y_2 e^{-iq_2 y_2} I(x_1, x_2, y_1, y_2) \\ &= \frac{(-i\lambda)^2}{2} D_F(p_1) D_F(p_2) D_F(q_1) D_F(q_2) \\ &\quad \times \underbrace{\int_{k_1} \int_{k_2} (2\pi)^4 \delta^4(p_1 + p_2 - k_1 - k_2) (2\pi)^4 \delta^4(k_1 + k_2 - q_1 - q_2) D_F(k_1) D_F(k_2)}_{=(2\pi)^4 \delta^4(p_1 + p_2 - q_1 - q_2) \int_k D_F(k) D_F(p_1 + p_2 - k)} \end{aligned} \quad (7.51)$$

By amputating the external propagators we obtain the contribution to the S-matrix element:

$$\begin{aligned} \text{FT} \{I(x_1, x_2, y_1, y_2)\}_{\text{amputated}} &= \\ &= (2\pi)^4 \delta^4(p_1 + p_2 - q_1 - q_2) \frac{(-i\lambda)^2}{2} \int_k D_F(k) D_F(p_1 + p_2 - k) \\ &= (2\pi)^4 \delta^4(p_1 + p_2 - q_1 - q_2) i\mathcal{M}. \end{aligned} \quad (7.52)$$

The δ -function reflects total momentum conservation; we already anticipated it when we defined the invariant amplitude via Eq. (7.48). Therefore, the result is simply

$$i\mathcal{M} = \frac{(-i\lambda)^2}{2} \int_k D_F(k) D_F(p_1 + p_2 - k). \quad (7.53)$$

From this expression we can read off the **Feynman rules for S-matrix elements**, which become extremely simple because all external propagators have disappeared. For an n -point function in a Φ^m theory at $\mathcal{O}(\lambda^k)$,

- draw n external points and k vertices with m ingoing lines, and connect all lines;
- write the propagators and vertices as

$$\begin{array}{c} \text{---} \blacktriangleleft \\ \text{P} \end{array} = D_F(p) = \frac{i}{p^2 - m_0^2 + i\varepsilon}, \quad \begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} = -i\lambda, \quad (7.54)$$

and impose momentum conservation at each vertex;

- integrate over all *loop* momenta $\int \frac{d^4k}{(2\pi)^4}$;
- divide by the symmetry factor of the diagram;
- set all external momenta onshell.

The two diagrams in Eq. (7.54) are the only elementary building blocks that we have at our disposal in a Φ^4 theory. In principle we can read them off directly from the Lagrangian:

$$\begin{aligned} S &= \int d^4x \left[\frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} m_0^2 \Phi^2 - \frac{\lambda}{m!} \Phi^m \right] \\ &\stackrel{p.I.}{\simeq} \int d^4x \left[-\frac{1}{2} \Phi (\square + m_0^2) \Phi - \frac{\lambda}{m!} \Phi^m \right]. \end{aligned} \quad (7.55)$$

After taking a Fourier transform of each field, the Klein-Gordon operator becomes the inverse tree-level propagator $D_F^{-1}(p) = p^2 - m_0^2 + i\varepsilon$ in momentum space, and the tree-level interaction vertex follows from removing the fields together with the combinatorial factor $4!$. In an extremely symbolic sense we could write the action (here for a Φ^4 theory) as

$$S = \begin{array}{c} \bullet \text{---} \bullet \\ -1 \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \text{---} \bullet \\ | \\ \bullet \end{array}$$

where the circles represent the fields $\Phi(x)$. Such a symbolic notation is indeed useful in the path-integral approach, where Green functions are obtained as functional derivatives of the classical action or the quantum effective action.

To summarize, the basic goal of a quantum field theory is to calculate the fully dressed n -point Green functions, including all quantum corrections, by starting from the tree-level expressions that are specified by the classical Lagrangian. These n -point functions are the quantities that enter scattering matrix elements from where we can extract observables.

8 Loops and renormalization

From the Feynman rules in the last section we know how to write down the diagrams that contribute to a given n -point function at some order in perturbation theory. Take for example the four-point function in ϕ^4 theory:

$$i\mathcal{M} = -i\lambda + \frac{(-i\lambda)^2}{2} \underbrace{\left[\int \frac{d^4k}{(2\pi)^4} D_F(k) D_F(p-k) + \text{perm.} \right]}_{=: \mathcal{A}(p)} + \mathcal{O}(\lambda^3). \quad (8.1)$$

Here, $-i\lambda$ is the tree-level vertex and $\mathcal{A}(p)$ with $p = p_1 + p_2$ the amputated 1-loop diagram in Eq. (7.49) that leads to Eq. (7.53):

$$\mathcal{A}(p) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_0^2 + i\epsilon} \frac{i}{(k-p)^2 - m_0^2 + i\epsilon}. \quad (8.2)$$

It depends on an external momentum p and we integrate over the loop momentum k . For $k^2 \rightarrow \infty$, the integral is proportional to d^4k/k^4 , and therefore the integral diverges logarithmically.

The question is: how can we actually calculate such integrals and isolate the divergences that they contain? And after doing so, what should we do with them? It will turn out that the structure of 1-loop integrals is the same independently of the theory we are interested in, so eventually we can take over the results directly to QED.

Feynman parameters. The first step is a convenient trick based on the formula

$$\int_0^1 dx \frac{1}{[xa + (1-x)b]^2} = -\frac{1}{a-b} \frac{1}{xa + (1-x)b} \Big|_0^1 = -\frac{1}{a-b} \left(\frac{1}{a} - \frac{1}{b} \right) = \frac{1}{ab}, \quad (8.3)$$

which we can also write in the form

$$\frac{1}{ab} = \int_0^1 dx \int_0^1 dy \delta(x+y-1) \frac{1}{(xa+yb)^2}, \quad (8.4)$$

where $x, y \in [0, 1]$ are called Feynman parameters. More generally,

$$\frac{1}{a_1 \dots a_n} = \int dx_1 \dots dx_n \delta\left(\sum_{i=1}^n x_i - 1\right) \frac{(n-1)!}{\left[\sum_{i=1}^n x_i a_i\right]^n}. \quad (8.5)$$

The structure of loop integrals is always that of Eq. (8.2), with one loop momentum k and one or several external momenta p_i , and possibly more than just two internal propagators. Let's evaluate the formula specifically for $a_i = (k + p_i)^2 - m_i^2 + i\epsilon$. In that case

$$\sum_i x_i a_i = \sum_i x_i (k^2 + p_i^2 + 2k \cdot p_i - m_i^2 + i\epsilon) = k^2 + \sum_i x_i (2k \cdot p_i + p_i^2 - m_i^2) + i\epsilon,$$

where we exploited the constraint $\sum_i x_i = 1$ that is imposed by the δ -function. Now, define a new loop momentum l via

$$l = k + \sum_i x_i p_i \quad \Rightarrow \quad l^2 = k^2 + 2 \sum_i x_i k \cdot p_i + \left(\sum_i x_i p_i \right)^2, \quad (8.6)$$

which leads to

$$\sum_i x_i a_i = l^2 - \underbrace{\left[\left(\sum_i x_i p_i \right)^2 - \sum_i x_i (p_i^2 - m_i^2) \right]}_{=: \Delta} + i\epsilon = l^2 - \Delta + i\epsilon. \quad (8.7)$$

The quantity Δ no longer depends on the loop momentum l . The expression (8.2) corresponds to $n = 2$; the resulting integrand only depends on l^2 :

$$\mathcal{A}(p) = - \int_0^1 dx \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - \Delta + i\epsilon)^2}. \quad (8.8)$$

In that case $p_1 = -p$, $p_2 = 0$ and $m_1 = m_2 = m_0$, and therefore $l = k - xp$ and

$$\Delta = x^2 p^2 - xp^2 + xm_0^2 + (1-x)m_0^2 = m_0^2 - x(1-x)p^2. \quad (8.9)$$

Wick rotation. The pole structure of $\mathcal{A}(p)$ is the same as that for a single propagator: when we split the integral $\int d^4 l = \int d^3 l \int dl_0$, the bracket in the denominator gives

$$l^2 - \Delta + i\epsilon = l_0^2 - (\mathbf{l}^2 + \Delta) + i\epsilon, \quad (8.10)$$

with the same Feynman prescription for the integration contour: integrate below the pole at negative l_0 and above the pole at positive l_0 . Since there are no further poles in the complex l_0 plane, we can equally deform the integration contour to follow the imaginary axis (Wick rotation) and define a **Euclidean momentum** l_E^μ :

$$l^0 = il_E^0, \quad \mathbf{l} = \mathbf{l}_E \quad \Rightarrow \quad l^2 = -(l_E^0)^2 - \mathbf{l}_E^2 = -l_E^2, \quad d^4 l = id^4 l_E. \quad (8.11)$$

The integral then becomes

$$\mathcal{A}(p) = -i \int_0^1 dx I_2^{(4)}, \quad I_2^{(4)} := \int \frac{d^4 l_E}{(2\pi)^4} \frac{1}{(l_E^2 + \Delta)^2}, \quad (8.12)$$

where the subscript ‘2’ is the power of the denominator and the superscript ‘(4)’ the number of spacetime dimensions. For general loop integrals we arrive at the formula

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{\prod_i [(k + p_i)^2 - m_i^2 + i\epsilon]} = i(-1)^n (n-1)! \int dx_1 \dots dx_n \delta\left(\sum_i x_i - 1\right) I_n^{(4)}, \quad (8.13)$$

with l defined in Eq. (8.6) and Δ in Eq. (8.7).

Regularization. Next, we want to calculate the integral $I_2^{(4)}$ explicitly. To do so, we write the four-dimensional integral as

$$d^4l_E = dl_E l_E^3 d\Omega_4 = \frac{1}{2} dl_E^2 l_E^2 d\Omega_4, \quad (8.14)$$

where $d\Omega_4$ is the four-dimensional unit sphere and $\int d\Omega_4 = 2\pi^2$. Hence we are left with a radial integral

$$I_2^{(4)} = \frac{1}{(4\pi)^2} \int_0^\infty dl_E^2 \frac{l_E^2}{(l_E^2 + \Delta)^2}, \quad (8.15)$$

which diverges logarithmically when $l_E^2 \rightarrow \infty$.

The idea is to make the integral finite by introducing some regulator, which will also isolate the components that diverge once the regulator is removed. There are several possible ways of regularization. We will discuss three of them here: a momentum cutoff, Pauli-Villars regularization, and dimensional regularization. There are also other well-established methods such as lattice regularization, proper-time regularization etc.

UV momentum cutoff. Since the divergence is produced by the UV momentum modes, the simplest strategy is to impose a hard cutoff: we do not integrate l_E^2 over the full momentum range but only up to a cutoff $l_E^2 < \Lambda^2$. Setting $l_E^2 = z$, the integral becomes:

$$\begin{aligned} \int_0^{\Lambda^2} dz \frac{z}{(z + \Delta)^2} &= \int_0^{\Lambda^2} dz \left[\frac{z + \Delta}{(z + \Delta)^2} - \frac{\Delta}{(z + \Delta)^2} \right] = \left[\ln(z + \Delta) + \frac{z}{z + \Delta} \right]_0^{\Lambda^2} \\ &= \ln \left(\frac{\Lambda^2 + \Delta}{\Delta} \right) + \frac{\Delta}{\Lambda^2 + \Delta} - 1 \xrightarrow{\Lambda \rightarrow \infty} \ln \frac{\Lambda^2}{\Delta}. \end{aligned} \quad (8.16)$$

In the context of QED we will later see that a cutoff regularization breaks gauge invariance, so it is not the most suitable method to use. In practice it is more convenient to use dimensional regularization or Pauli-Villars regularization which both preserve gauge invariance.⁵

Pauli-Villars regularization. The idea of Pauli-Villars regularization is to modify one of the propagators in the loop integral so that the integrand vanishes faster in the ultraviolet. To do so, we start from the original expression (8.2), where we subtract another propagator with a large mass $\sqrt{m_0^2 + \Lambda^2}$:

$$\frac{1}{k^2 - m_0^2} \rightarrow \frac{1}{k^2 - m_0^2} - \frac{1}{k^2 - m_0^2 - \Lambda^2} = \frac{1}{k^2 - m_0^2} \frac{1}{1 - \frac{k^2 - m_0^2}{\Lambda^2}}. \quad (8.17)$$

Therefore, the propagator now vanishes as $\sim 1/k^4$ for $k^2 \rightarrow \infty$, and the integrand with a power $\sim 1/k^6$. The remaining steps up to Eq. (8.15) go through as before, but we

⁵One should keep in mind, however, that in the course of a numerical evaluation of loop integrals, where the momentum integration becomes a discretized sum, one always introduces a hard cutoff because a computer cannot integrate up to infinity. In that case one has to be especially careful about potential gauge artifacts.

have to subtract $I_2^{(4)} - I_2'^{(4)}$, where $I_2'^{(4)}$ is obtained from setting

$$\begin{aligned} p_1 &= -p, & m_1 &= \sqrt{m_0^2 + \Lambda^2}, & \Rightarrow & \Delta' = \Delta + x\Lambda^2 \\ p_2 &= 0, & m_2 &= m_0 \end{aligned} \quad (8.18)$$

in the Feynman parameter representation. Then we get

$$\int_0^\infty dl_E^2 \left[\frac{l_E^2}{(l_E^2 + \Delta)^2} - \frac{l_E^2}{(l_E^2 + \Delta + x\Lambda^2)^2} \right] \doteq \ln \left(1 + \frac{x\Lambda^2}{\Delta} \right) \quad (8.19)$$

which for $\Lambda \rightarrow \infty$ diverges again logarithmically.

Dimensional regularization. The most common regularization method in the context of perturbation theory is dimensional regularization. Here the idea is to first calculate the integral in d dimensions and take the limit $d \rightarrow 4$ in the end. We write

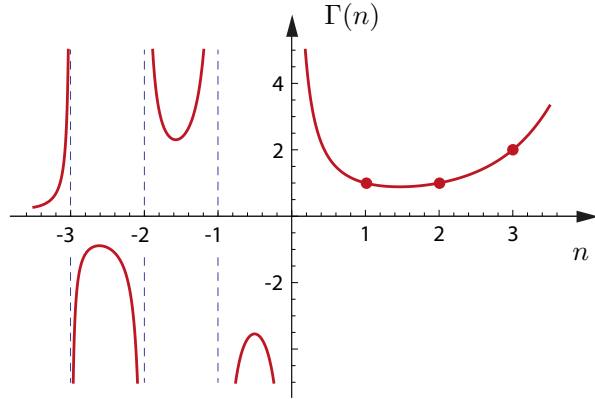
$$I_2^{(d)} = \frac{1}{M^{d-4}} \int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta^2)^2}, \quad (8.20)$$

where the factor M is an arbitrary mass scale that we introduced to ensure that the integral remains dimensionless also in d spacetime dimensions. Its origin is the dimension of the coupling constant in front of the integral: for a ϕ^4 theory in four dimensions, λ is dimensionless but this is no longer the case for arbitrary d . The volume integral becomes

$$d^d l_E = dl_E l_E^{d-1} d\Omega_d = \frac{1}{2} dl_E^2 (l_E^2)^{\frac{d}{2}-1} d\Omega_d, \quad \int d\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}, \quad (8.21)$$

where $d\Omega_d$ is the unit sphere in d dimensions. $\Gamma(n)$ is the Gamma function; let us recall a few of its properties:

- $\Gamma(n) = \int_0^\infty dx x^{n-1} e^{-x}$,
- $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}_+$,
- $\Gamma(n)$ has poles at $n = 0, -1, -2, \dots$
- $\Gamma(n+1) = n\Gamma(n)$,
- $\Gamma'(1) = -\gamma = -0.5772\dots$ is the Euler-Mascheroni constant.



It is easy to prove the result (8.21) for $\int d\Omega_d$:

$$\begin{aligned} (\sqrt{\pi})^d &= \left(\int_{-\infty}^{\infty} dx e^{-x^2} \right)^d = \int d^d x e^{-\sum_{i=1}^d x_i^2} \\ &= \frac{1}{2} \int dx^2 (x^2)^{d/2-1} e^{-x^2} \int d\Omega_d = \frac{1}{2} \Gamma\left(\frac{d}{2}\right) \int d\Omega_d. \end{aligned} \quad (8.22)$$

Now, take the integral (8.20) and insert Eq. (8.21):

$$I_2^{(d)} = \frac{1}{M^{d-4}} \frac{\pi^{d/2}}{(2\pi)^d} \frac{1}{\Gamma(\frac{d}{2})} \int_0^\infty dl_E^2 \frac{(l_E^2)^{d/2-1}}{(l_E^2 + \Delta)^2}. \quad (8.23)$$

With the substitution

$$z = \frac{\Delta}{l_E^2 + \Delta} \quad \Rightarrow \quad dz = -dl_E^2 \frac{\Delta}{(l_E^2 + \Delta)^2}, \quad l_E^2 = \frac{\Delta}{z} (1 - z) \quad (8.24)$$

we can transform it into

$$I_2^{(d)} = \frac{1}{M^{d-4}} \frac{1}{(4\pi)^{d/2}} \frac{1}{\Gamma(\frac{d}{2})} \left(\frac{1}{\Delta}\right)^{2-d/2} \underbrace{\int_0^1 dz z^{1-d/2} (1-z)^{d/2-1}}_{= \mathcal{B}(2-\frac{d}{2}, \frac{d}{2}) = \frac{\Gamma(2-\frac{d}{2}) \Gamma(\frac{d}{2})}{\Gamma(2)}}. \quad (8.25)$$

We expressed the remaining integral through Euler's Beta function

$$\mathcal{B}(m, n) = \int_0^1 dx x^{m-1} (1-x)^{n-1} = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}, \quad (8.26)$$

so that we arrive at the result

$$I_2^{(d)} = \frac{1}{M^{d-4}} \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \left(\frac{1}{\Delta}\right)^{2-d/2}. \quad (8.27)$$

This expression diverges for $d = 4, 6, 8, \dots$ but is otherwise well-defined, even if d is non-integer. Hence, we can use it as a *definition* of the original integral for non-integer dimensions.

In the final step we set $d = 4 - \varepsilon$,

$$I_2^{(d)} = M^\varepsilon \frac{\Gamma(\frac{\varepsilon}{2})}{(4\pi)^{2-\varepsilon/2}} \left(\frac{1}{\Delta}\right)^{\varepsilon/2} = \frac{\Gamma(\frac{\varepsilon}{2})}{(4\pi)^2} \left(\frac{4\pi M^2}{\Delta}\right)^{\varepsilon/2}, \quad (8.28)$$

and expand the expression around $\varepsilon = 0$. Using $x^{\varepsilon/2} = e^{\frac{\varepsilon}{2} \ln x} = 1 + \frac{\varepsilon}{2} \ln x + \mathcal{O}(\varepsilon^2)$ and $\Gamma(\frac{\varepsilon}{2}) = \frac{2}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon)$, we find

$$I_2^{(d)} = \frac{1}{(4\pi)^2} \left[\frac{2}{\varepsilon} - \gamma + \ln \left(\frac{4\pi M^2}{\Delta} \right) + \mathcal{O}(\varepsilon) \right]. \quad (8.29)$$

The integral has a part $\sim 1/\varepsilon$ that diverges for $\varepsilon \rightarrow 0$, and a remainder that is finite and depends on M , which is completely arbitrary because it was only introduced for dimensional reasons. In principle we could also combine the finite parts since $-\gamma = \ln e^{-\gamma}$ and write

$$-\gamma + \ln \left(\frac{4\pi M^2}{\Delta} \right) = \ln \left(\frac{4\pi M^2 e^{-\gamma}}{\Delta} \right) = \ln \frac{\widetilde{M}^2}{\Delta}. \quad (8.30)$$

The finite parts have formally the same structure as for cutoff and Pauli-Villars regularization, because also in those cases we can always introduce a mass scale \widetilde{M} such that for $\Lambda \rightarrow \infty$

$$\ln \frac{x\Lambda^2}{\Delta} = \ln \frac{x\Lambda^2}{\widetilde{M}^2} + \ln \frac{\widetilde{M}^2}{\Delta}. \quad (8.31)$$

The divergent terms differ, however: they may diverge logarithmically with $\ln \Lambda^2$, or with $1/\varepsilon$ as in dimensional regularization.

In complete analogy one can also work out the following integrals:

$$\begin{aligned} I_n^{(d)} &= \int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2}}, \\ \widetilde{I}_n^{(d)} &= \int \frac{d^d l_E}{(2\pi)^d} \frac{l_E^2}{(l_E^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - \frac{d}{2} - 1}. \end{aligned} \quad (8.32)$$

In summary, the expression for $\mathcal{A}(p)$ in Eq. (8.1), using Eqs. (8.12) and (8.29), becomes

$$\mathcal{A}(p) = -i \int_0^1 dx I_2^{(4)} = -\frac{i}{(4\pi)^2} \lim_{\varepsilon \rightarrow 0} \int_0^1 dx \left[\frac{2}{\varepsilon} - \gamma + \ln \left(\frac{4\pi M^2}{\Delta} \right) \right]. \quad (8.33)$$

A common feature of all regularization methods is that they always introduce a **scale** M in the theory, which remains there even if we formally remove the divergent terms. This new scale dependence has profound consequences: even if the mass parameter in the Lagrangian is zero and the classical theory is scale invariant, the renormalized quantum field theory is *not* because in the process of regularization we have picked up a scale. Classical symmetries that are broken at the quantum level are called **anomalous**, so this effect is also called the ‘anomalous breaking of scale invariance’.

Renormalization. So we can calculate loop diagrams explicitly by introducing some regulator, and we can separate the finite parts from the divergent ones. The ultimate question is: what should we do with the divergences? Should we simply throw them away, and if yes, how would that make any sense? Surprisingly enough, this is indeed what eventually has to happen, but there is a deeper underlying reason which can be understood in the course of renormalization. The idea is the following: let’s interpret all fields, masses and couplings that appear in the Lagrangian as ‘bare’ and unphysical, and write the Lagrangian as

$$\mathcal{L} = \frac{1}{2} \partial_\mu \Phi_B \partial^\mu \Phi_B - \frac{1}{2} m_B^2 \Phi_B^2 - \frac{\lambda_B}{4!} \Phi_B^4 \stackrel{\text{p.I.}}{=} -\frac{1}{2} \Phi_B (\square + m_B^2) \Phi_B - \frac{\lambda_B}{4!} \Phi_B^4 \quad (8.34)$$

with a subscript ‘B’ for bare. Now define a **renormalized** field Φ , renormalized mass m and renormalized coupling λ by

$$\Phi_B = Z_\phi^{1/2} \Phi, \quad m_B^2 = Z_m m^2, \quad \lambda_B = Z_\lambda \lambda, \quad (8.35)$$

where we introduced three **renormalization constants** Z_ϕ , Z_m and Z_λ . They are, as of now, undetermined and potentially divergent. Consequently, the Lagrangian takes

the form⁶

$$\mathcal{L} = -\frac{1}{2} Z_\phi \Phi (\square + Z_m m^2) \Phi - Z_\lambda Z_\phi^2 \frac{\lambda}{4!} \Phi^4. \quad (8.36)$$

Since we can read off the tree-level propagators and vertices from the Lagrangian, the renormalization constants will also enter in their Feynman rules. We will call them the *renormalized tree-level* propagator and vertex:

$$\begin{array}{l} \text{---} \overleftarrow{\text{P}} \text{---} \\ \text{---} \times \text{---} \end{array} \quad \begin{array}{l} D_0(p) = \frac{i}{Z_\phi (p^2 - Z_m m^2)} \quad \Leftrightarrow \quad iD_0^{-1}(p) = Z_\phi (p^2 - Z_m m^2), \\ \Gamma_0(\{p_i\}) = -i\lambda Z_\lambda Z_\phi^2. \end{array} \quad (8.37)$$

The $i\epsilon$ prescription is still intact but we drop it for brevity.

Consider now the **full 1PI Green functions** of the theory. The set of all 1PI functions defines the quantum field theory completely because the effective action can be expressed by them (we might return to this at some later point). We have seen that we can reconstruct the propagator from its 1PI counterpart, Eq. (7.42), and generally this is true for all S-matrix elements: the connected, amputated S-matrix elements can be expressed in terms of 1PI Green functions together with dressed propagator insertions. The 1PI n-point functions are also convenient for the discussion of renormalization as we will see shortly. If we denote the full propagator by $D(p)$, then it is related to the 1PI self-energy via

$$D(p) = D_0 + D_0 \frac{\Sigma}{i} D_0 + \dots = D_0 (1 + i\Sigma D) \quad \Rightarrow \quad iD^{-1} = iD_0^{-1} - \Sigma, \quad (8.38)$$

and so we can generally write

$$\begin{array}{l} \text{---} \bullet \text{---} \\ \text{---} \times \text{---} \end{array} \quad \begin{array}{l} iD^{-1}(p) = Z_\phi (p^2 - Z_m m^2) - \Sigma(p), \\ \Gamma(\{p_i\}) = -i\lambda Z_\lambda Z_\phi^2 + i\Omega(\{p_i\}). \end{array} \quad (8.39)$$

$\Sigma(p)$ defines the self-energy as before, and its analogue for the four-point function is Ω : it contains all 1PI loop diagrams that we can draw order by order in perturbation theory. In terms of Feynman diagrams:

$$\begin{array}{l} \text{---} \bullet \text{---}^{-1} \\ \text{---} \times \text{---} \end{array} = \begin{array}{l} \text{---}^{-1} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} + \dots \\ \text{---} \times \text{---} + \text{---} \text{---} \text{---} + \dots \end{array} \quad (8.40)$$

In principle the list goes on for the six-point function, eight-point function, etc.,

$$\begin{array}{l} \text{---} \bullet \text{---} \\ \text{---} \times \text{---} \end{array} \quad \dots \quad (8.41)$$

⁶This way of discussing renormalization is also called ‘renormalized perturbation theory’. The alternative is ‘bare perturbation theory’ which is completely equivalent but somewhat more confusing, so we will not discuss it here.

except that they do not have tree-level contributions but start off with loop diagrams right away. In ϕ^4 theory there are also no n -point functions with an odd number of legs; this is due to the invariance of the Lagrangian under $\phi \rightarrow -\phi$.

The idea is now that the full propagator should have a pole at $p^2 = m^2$, where it corresponds to a free particle with mass m . Likewise, the full vertex should become a free vertex if its external legs are onshell:

$$D(p) \xrightarrow{p^2=m^2} \frac{i}{p^2 - m^2}, \quad \Gamma(\{p_i\}) \xrightarrow{p_i^2=m^2} -i\lambda. \quad (8.42)$$

Here, m and λ are the physical, measurable mass and coupling constant of the theory. Actually these **renormalization conditions** are completely arbitrary, so it makes sense to generalize them to some arbitrary **renormalization point** $p^2 = \mu^2$. This is especially practical in theories where the propagator does *not* have a Källén-Lehmann representation. An example is QCD, where there are no free quarks due to confinement. Hence we demand

$$iD^{-1}(p) \Big|_{p^2=\mu^2} \stackrel{!}{=} p^2 - m^2, \quad \frac{d}{dp^2} iD^{-1}(p) \Big|_{p^2=\mu^2} \stackrel{!}{=} 1, \quad \Gamma(\{p_i\}) \Big|_{p_i^2=\mu^2} \stackrel{!}{=} -i\lambda. \quad (8.43)$$

The first condition fixes the ‘pole position’ through the mass m (which is a true pole only if $\mu = m$), the second sets the residue at the pole, and the third fixes the coupling constant. Now let’s insert this into Eq. (8.39). If we abbreviate

$$\Sigma(p) \Big|_{p^2=\mu^2} = \Sigma_\mu, \quad \frac{d}{dp^2} \Sigma(p) \Big|_{p^2=\mu^2} = \Sigma'_\mu, \quad \Omega(\{p_i\}) \Big|_{p_i^2=\mu^2} = \Omega_\mu \quad (8.44)$$

we arrive at

$$\begin{aligned} Z_\phi(\mu^2 - Z_m m^2) - \Sigma_\mu = \mu^2 - m^2 &\Rightarrow Z_\phi Z_m = 1 + \frac{\mu^2 \Sigma'_\mu - \Sigma_\mu}{m^2}, \\ Z_\phi - \Sigma'_\mu = 1 &\Rightarrow Z_\phi = 1 + \Sigma'_\mu, \\ -i\lambda Z_\lambda Z_\phi^2 + i\Omega_\mu = -i\lambda &\Rightarrow Z_\phi^2 Z_\lambda = 1 + \frac{\Omega_\mu}{\lambda}. \end{aligned} \quad (8.45)$$

These conditions determine the three renormalization constants: at lowest order perturbation theory they are all equal to one, whereas at higher orders they pick up loop contributions from Σ_μ , Σ'_μ and Ω_μ which have divergent and finite parts. Hence their generic structure is of the form

$$Z_i(\lambda, m, \epsilon) = 1 + \sum_{k=1}^{\infty} c_k(\lambda, m, \epsilon) \lambda^k, \quad (8.46)$$

with divergent coefficients c_k . On the other hand, when we substitute this back into Eq. (8.39) we find

$$\begin{aligned} iD^{-1}(p) &= (1 + \Sigma'_\mu) p^2 - m^2 - \mu^2 \Sigma'_\mu + \Sigma_\mu - \Sigma(p) \\ &= p^2 - m^2 - (\Sigma(p) - \Sigma_\mu) + (p^2 - \mu^2) \Sigma'_\mu, \\ \Gamma(\{p_i\}) &= -i\lambda + i(\Omega(\{p_i\}) - \Omega_\mu). \end{aligned} \quad (8.47)$$

The crucial point is that by means of the subtraction at the renormalization point the divergences *cancel* in the renormalized Green functions. Therefore, the renormalized n-point functions are finite!

Let's have a look at a concrete example, namely the one-loop contribution to the four-point function. We have worked out its structure earlier; the result in dimensional regularization was Eq. (8.33):

$$\Omega(\{p_i\}) = \frac{\lambda^2}{2} \int_0^1 dx I_2^{(4)} = \frac{\lambda^2}{2} \frac{1}{(4\pi)^2} \int_0^1 dx \left[\frac{2}{\epsilon} - \gamma + \ln \frac{4\pi M^2}{\Delta} \right], \quad (8.48)$$

with $\Delta = m_B^2 - x(1-x)p^2$ and $p = p_1 + p_2$, plus the two permutations which we do not write explicitly. In principle, by means of the Feynman rules (8.37) the diagram picks up an additional prefactor

$$\frac{Z_\lambda^2 Z_\phi^4}{Z_\phi^2} = Z_\lambda^2 Z_\phi^2 = 1 + \mathcal{O}(\lambda), \quad (8.49)$$

but since the correction comes with powers of the coupling constant it will only contribute at higher orders in perturbation theory, so we can ignore it in the one-loop result. For simplicity we renormalize the four-point function at $p^2 = (p_1 + p_2)^2 = \mu^2$. Observe that the subtraction cancels the divergent piece $\sim 1/\epsilon$:

$$\begin{aligned} \Omega(\{p_i\}) - \Omega_\mu &= \frac{\lambda^2}{2} \frac{1}{(4\pi)^2} \int_0^1 dx \ln \frac{\Delta_\mu}{\Delta} = \frac{\lambda^2}{2} \frac{1}{(4\pi)^2} \int_0^1 dx \ln \frac{m_B^2 - x(1-x)\mu^2}{m_B^2 - x(1-x)p^2} \\ &= \frac{\lambda^2}{2} \frac{1}{(4\pi)^2} \int_0^1 dx \ln \frac{m^2 - x(1-x)\mu^2}{m^2 - x(1-x)p^2}. \end{aligned} \quad (8.50)$$

In the last step we have used that $m_B^2 = Z_m m^2 = m^2 + \mathcal{O}(\lambda)$, so the correction will also only appear at higher orders and to lowest order we can set $m_B = m$. The resulting expression depends on the renormalized mass m and coupling λ . It is finite, but in turn it depends now on the arbitrary renormalization point μ .

Counterterms. It is customary to write the renormalization constants as

$$Z_\phi = 1 + \delta Z_\phi, \quad Z_m Z_\phi = 1 + \frac{\delta m^2}{m^2}, \quad Z_\lambda Z_\phi^2 = 1 + \frac{\delta \lambda}{\lambda}. \quad (8.51)$$

In that way the Lagrangian (8.34) can be split into a piece that depends only on renormalized quantities, plus a counterterm that includes the new 'renormalization constants' δZ_ϕ , δm^2 and $\delta \lambda$:

$$\mathcal{L} = -\frac{1}{2} \Phi (\square + m^2) \Phi - \frac{\lambda}{4!} \Phi^4 - \frac{1}{2} \Phi (\delta Z_\phi \square + \delta m^2) \Phi - \frac{\delta \lambda}{4!} \Phi^4. \quad (8.52)$$

The counterterms can be interpreted as new tree-level propagators and vertices with corresponding Feynman rules. This is especially convenient for calculating higher loops,

because eventually it would become hard to keep track of the Z_i factors in front of the integrals from lower orders in perturbation theory (which we can ignore for one-loop graphs). Instead, one must now systematically add diagrams with ‘counter’ propagators and vertices. The expressions (8.39) for the full 1PI Green functions become

$$\begin{aligned}
 \text{---}\bullet\text{---} & \quad iD^{-1}(p) = p^2 - m^2 - \Sigma(p) + \delta Z_\phi p^2 - \delta m^2, \\
 \bullet & \quad i\Gamma(\{p_i\}) = \lambda - \Omega(\{p_i\}) + \delta\lambda,
 \end{aligned} \tag{8.53}$$

i.e., the new renormalization constants can be directly identified with the counterterms that cancel the singularities. If we apply our earlier renormalization conditions and compare Eq. (8.51) with (8.45) we find

$$\delta Z_\phi = \Sigma'_\mu, \quad \delta m^2 = \mu^2 \Sigma'_\mu - \Sigma_\mu, \quad \delta\lambda = \Omega_\mu. \tag{8.54}$$

Renormalization schemes. The examples discussed so far highlight some general features of renormalization:

- If a given theory contains a finite number of renormalization constants Z_i (three in ϕ^4 theory), we must specify equally many renormalization conditions to determine them. This in turn removes all UV divergences from the theory. We will provide more detailed arguments below.
- All physical quantities are independent of ϵ and Z_i and they are finite. The Lagrangian \mathcal{L} itself is divergent, but this is irrelevant because it is not an observable.
- The mass $m(\mu)$ and coupling $\lambda(\mu)$ depend now on the renormalization point μ , where they are specified as an external input. That is, they are parameters of the theory and can no longer be determined *within* the theory — they must be taken from experiment.

In QED we can use **onshell renormalization** with $\mu^2 = m^2$. The electron propagator has a pole at $p^2 = m^2$, where m is the physical mass of the electron. The photon is massless, so its propagator has a pole at $q^2 = 0$. This is where one can match the coupling constant (the electron charge) with experiment, because two infinitely separated charges correspond to a propagator evaluated at $q^2 = 0$. On the other hand, onshell renormalization doesn't work in QCD because there are no free quarks and gluons due to confinement. As a consequence, the quark masses and the coupling have to be specified at some suitable renormalization scale where theory predictions can be compared to experiment.⁷

The arbitrariness in the specification of $m(\mu)$ and $\lambda(\mu)$ is reflected in the **renormalization scheme**. Imposing *overall* renormalization conditions of the form (8.43) on the Green functions defines a momentum subtraction (MOM) scheme. This is convenient for nonperturbative calculations since at no point in the previous discussion

⁷This scale should also be spacelike ($\mu^2 < 0$ in Minkowski conventions) to avoid branch-cut singularities that appear in the loop diagrams. High-energy scattering experiments with hadrons probe the domain of large spacelike momenta of internal quarks and gluons, which is also where the QCD coupling is small and perturbation theory applicable.

we *needed* to resort to a perturbative expansion: Eqs. (8.39) can be equally viewed as Dyson-Schwinger equations (cf. Eq. (7.46)) which are nonperturbative and exact. Alternatively, one can also explicitly subtract only the divergent terms order by order in perturbation theory, such as the one $\sim 1/\epsilon$ in Eq. (8.48), which defines the MS scheme (minimal subtraction). In that case our definition of the renormalization scale μ is no longer available; instead, the scale $M \equiv \mu$ takes its place as it doesn't get cancelled by the subtraction anymore. (In the MOM scheme, we have essentially traded the dependence on M by a dependence on μ .) Another possibility is to subtract not only the divergences but *all* terms that are not explicitly dependent on $M \equiv \mu$; this defines the $\overline{\text{MS}}$ scheme (modified minimal subtraction).

As a consequence, the masses and couplings depend not only on the renormalization point but also on the renormalization scheme, and the different schemes are related to each other by finite constants:

$$\begin{aligned} m(\mu)_{\text{MOM}} &\leftrightarrow m(\mu)_{\text{MS}} &\leftrightarrow m(\mu)_{\overline{\text{MS}}} &\leftrightarrow \dots \\ \lambda(\mu)_{\text{MOM}} &\leftrightarrow \lambda(\mu)_{\text{MS}} &\leftrightarrow \lambda(\mu)_{\overline{\text{MS}}} &\leftrightarrow \dots \end{aligned} \quad (8.55)$$

The Green functions themselves depend on the renormalization point μ , but they are independent of the scheme. For example:

$$D(p, \mu, m(\mu)_{\text{MOM}}, \lambda(\mu)_{\text{MOM}}) = D(p, \mu, m(\mu)_{\overline{\text{MS}}}, \lambda(\mu)_{\overline{\text{MS}}}) = \dots \quad (8.56)$$

The invariance of measurable quantities under a change of μ and different renormalization schemes leads to the concept of the **renormalization group**.

As an example, consider the 1PI four-point function and write it with counterterms as in Eq. (8.53):

$$\begin{aligned} i\Gamma(p) &= \lambda - \Omega(p) + \delta\lambda = \lambda - \frac{\lambda^2}{32\pi^2} \int dx \left[\frac{2}{\epsilon} - \gamma + \ln \frac{4\pi\mu^2}{\Delta} \right] + \delta\lambda \\ &= \lambda - \frac{\lambda^2}{32\pi^2} \left[\frac{2}{\epsilon} - \gamma + \ln \frac{4\pi\mu^2}{m^2} - \int dx \ln \left(1 - x(1-x) \frac{p^2}{m^2} \right) \right] + \delta\lambda. \end{aligned} \quad (8.57)$$

For simplicity we ignore again the contribution from the two permuted diagrams, so the expression depends only on the s -channel momentum $p = p_1 + p_2$. In the MOM scheme we impose the condition

$$i\Gamma(p)_{p^2=\mu^2} \stackrel{!}{=} \lambda_{\text{MOM}}, \quad (8.58)$$

which leads to

$$\delta\lambda_{\text{MOM}} = \frac{\lambda_{\text{MOM}}^2}{32\pi^2} \left[\frac{2}{\epsilon} - \gamma + \ln \frac{4\pi\mu^2}{m^2} - \int dx \ln \left(1 - x(1-x) \frac{\mu^2}{m^2} \right) \right]. \quad (8.59)$$

In the MS and $\overline{\text{MS}}$ scheme we do not impose such a condition but instead subtract terms by hand. In MS we would only subtract the divergent term, whereas in $\overline{\text{MS}}$ we also subtract the remaining μ -independent terms:

$$\delta\lambda_{\text{MS}} = \frac{\lambda_{\text{MS}}^2}{32\pi^2} \frac{2}{\epsilon}, \quad \delta\lambda_{\overline{\text{MS}}} = \frac{\lambda_{\overline{\text{MS}}}^2}{32\pi^2} \left[\frac{2}{\epsilon} - \gamma + \ln 4\pi \right]. \quad (8.60)$$

In any case, whatever we decide to do cannot change the four-point function, which must remain the same. For example evaluated at the renormalization point:

$$i\Gamma(p)_{p^2=\mu^2} = \lambda_{\text{MOM}} = \lambda_{\overline{\text{MS}}} - \frac{\lambda_{\overline{\text{MS}}}^2}{32\pi^2} \left[\ln \frac{\mu^2}{m_{\overline{\text{MS}}}^2} - \int dx \ln \left(1 - x(1-x) \frac{\mu^2}{m_{\overline{\text{MS}}}^2} \right) \right], \quad (8.61)$$

which gives us the relation between λ_{MOM} and $\lambda_{\overline{\text{MS}}}$.

Renormalizability. So far we have only considered one explicit diagram. Do the singularities *always* cancel? Let's consider the action for a generic ϕ^p theory:

$$S = - \int d^4x \left[\frac{1}{2} \Phi (\square + m^2) \Phi + \frac{\lambda}{p!} \Phi^p \right], \quad (8.62)$$

where we suppress the renormalization constants for simplicity. Now count the mass dimensions of the quantities that appear in the action:

$$[S] = 0 \quad \Rightarrow \quad [\mathcal{L}] = 4, \quad [\Phi] = 1, \quad [\Phi^p] = p, \quad [\lambda] = 4 - p. \quad (8.63)$$

From here we can infer the dimensions of the 1PI Green functions in momentum space:

$$\begin{aligned} \Gamma_2 &= \text{---} \bullet \text{---} = p^2 - m^2 + \dots \quad \Rightarrow \quad [\Gamma_2] = 2, \\ \Gamma_4 &= \text{---} \bullet \text{---} \stackrel{\phi^4}{=} -i\lambda + \dots \quad \Rightarrow \quad [\Gamma_4] = 0, \\ \Gamma_6 &= \text{---} \bullet \text{---} \stackrel{\phi^6}{=} -i\lambda + \dots \quad \Rightarrow \quad [\Gamma_6] = -2. \end{aligned} \quad (8.64)$$

Remember from Eq. (7.33) that the tree-level vertex is always of the form $-i\lambda$ as long as λ is the corresponding ϕ^4 , ϕ^6 , ... coupling constant. That is, in a ϕ^4 theory the six-point function does not have a tree-level term, in a ϕ^6 theory the four-point function does not have a tree-level term, etc. In any case, the dimension of Γ_n is always the same independently of p , because it is already determined by $-i\lambda$:

$$[\Gamma_n] = 4 - n. \quad (8.65)$$

On the other hand, we can also count the dimension of a given n -point function by going into some order in perturbation theory. In that case, we would count the number of loops L (each comes with dimension four), the number of internal propagators I (each comes with dimension -2), and the number of vertices (where each has dimension $[\lambda]$). Therefore:

$$[\Gamma_n] = 4L - 2I + [\lambda]V. \quad (8.66)$$

For example in ϕ^4 theory, where $[\lambda] = 0$:

$$\begin{array}{ccc} \text{---} \bigcirc \text{---} & \text{---} \triangle \text{---} & \text{---} \bigcirc \triangle \text{---} \\ L = 1 & L = 1 & L = 2 \\ I = 2 & I = 3 & I = 5 \\ \Rightarrow [\Gamma_4] = 0 & \Rightarrow [\Gamma_6] = -2 & \Rightarrow [\Gamma_6] = -2 \end{array} \quad (8.67)$$

Obviously this is consistent.

Now, the quantity $D = 4L - 2I$ also tells us how badly divergent a given diagram will be: if the number of loops L beats the number of propagators I it will diverge; if there are many propagators in a loop it will converge. D is called the **superficial degree of divergence**: if $D < 0$ the diagram converges, if $D \geq 0$ it will diverge. The

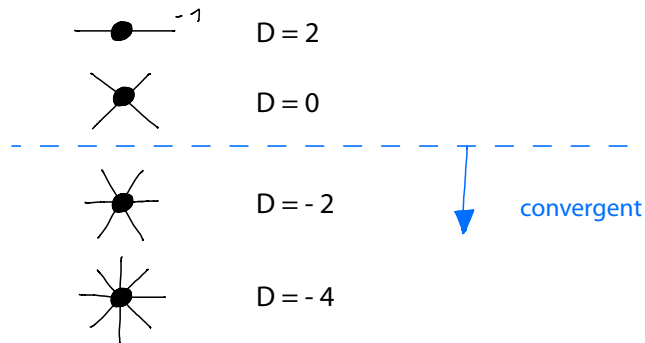
first diagram above has $D = 0$ and diverges logarithmically. The second has $D = -2$ and is convergent; the third has $D = -2$ but unfortunately it is still divergent because it contains a divergent subdiagram (the one on the left). Hence the name ‘superficial’ degree of divergence:

- a diagram with $D \geq 0$ can still be finite due to cancellations,
- a diagram with $D < 0$ can be divergent if it contains divergent subdiagrams,
- tree-level diagrams have $D = 0$ but they are finite.

Let’s ignore these subtleties for a moment and assume that D counts the actual degree of divergence. From Eq. (8.66) we can determine it as

$$D = [\Gamma_n] - [\lambda] V. \quad (8.68)$$




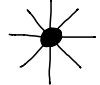
The mass dimension $[\Gamma_n]$ is fixed and does not depend on the order in perturbation theory, which is determined by V . However, D depends on V — it rises or falls with higher orders depending on the mass dimension of the coupling $[\lambda]$. Take ϕ^4 theory, where $[\lambda] = 0$ and D is independent of V :

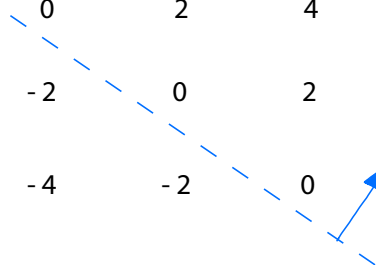


Therefore, there are only two divergent Green functions in ϕ^4 theory: the inverse propagator and the four-point function. Those are exactly the ones with a tree-level term in the Lagrangian; they are also called the **primitively divergent Green functions** .

One can indeed show that the analysis goes through in general, also for divergent subdiagrams, which is known as the **BPHZ theorem** (Bogoliubov, Parasiuk, Hepp, Zimmermann). The reason is that the Z_i factors in front of the diagrams (which we can neglect at one-loop) cancel the divergences at higher orders. Take for example the two diagrams on the right in Eq. (8.67): both contribute to the six-point function, one with $V = 3$ and the other with $V = 4$. The $V = 3$ diagram carries factors $Z = 1 + \delta Z$, where δZ contributes at higher order to the $V = 4$ graph. The sum of all contributions at a given order cancels the divergences. Here it is especially useful to employ the counterterm language, because the subdivergences will cancel with the counterterms at each order in perturbation theory.

On the other hand, the same analysis for ϕ^6 theory gives us:

	V = 0	V = 1	V = 2
 ⁻¹	2	4	6
	0	2	4
	-2	0	2
	-4	-2	0

divergent 

In other words, if we go high enough in perturbation theory eventually *every* Green function will diverge!

This leads to the notion of **renormalizability**: a theory is renormalizable if only a *finite* number of Green functions have $D \geq 0$, so that only a finite number of renormalization conditions are necessary to remove the divergences from the theory. From Eq. (8.68) this is equivalent to the following statement:

$$\text{A theory is renormalizable if } [\lambda] \geq 0.$$

That is, the coupling must be either dimensionless or have a positive mass dimension (in the latter case the theory is called *super-renormalizable*). A non-renormalizable theory has a coupling with negative mass dimension: in that case every Green function eventually becomes divergent. Here we would need new renormalization conditions at each order in perturbation theory, and eventually infinitely many, so we must specify infinitely many constants from outside. The theory thereby loses its predictive power.

The good news is that we can read off a theory's renormalizability directly from its Lagrangian: we just need to look at the mass dimension of the coupling constant. For a scalar ϕ^p theory only ϕ^3 and ϕ^4 interactions are renormalizable whereas those with $p > 4$ are not. Renormalizability restricts the possible forms of interactions dramatically!

9 Cross sections and decay rates

[To come]

10 QED

[To come]

Lagrangian and gauge invariance.

Feynman rules for fermions.

Feynman rules for photons.

Tree-level scattering processes.

Coulomb potential.

Cross sections.

11 Renormalization of QED

Renormalization constants. We have already discussed the underlying principles of renormalization in the context of a scalar field. This saves us from the trouble of going through the same steps all over again: we don't need to calculate loop diagrams in QED only to realize that they diverge and then figure out what to do about it, because with a few adaptations we can take over the ideas from the scalar theory.

Once again, we interpret all quantities in the Lagrangian as bare and unphysical,

$$\begin{aligned} \mathcal{L}_{\text{QED}} \stackrel{\text{p.I.}}{\simeq} & \bar{\psi}_B (i\not{\partial} - m_B) \psi_B + g_B \bar{\psi}_B \not{A}_B \psi_B \\ & + \frac{1}{2} A_B^\mu (\square g_{\mu\nu} - \partial_\mu \partial_\nu) A_B^\nu + \frac{\lambda_B}{2} A_B^\mu \partial_\mu \partial_\nu A_B^\nu, \end{aligned} \quad (11.1)$$

and define their renormalized counterparts by

$$\psi_B = Z_\psi^{1/2} \psi, \quad A_B = Z_A^{1/2} A, \quad m_B = Z_m m, \quad g_B = Z_g g, \quad \lambda_B = Z_\lambda \lambda. \quad (11.2)$$

In principle there are five renormalization constants, but we will later see that gauge invariance relates two of them via Ward identities:

$$Z_g Z_A^{1/2} = 1, \quad Z_\lambda Z_A = 1. \quad (11.3)$$

Hence, there are just three independent renormalization constants: Z_ψ , Z_A and Z_m . The resulting Lagrangian takes the form⁸

$$\begin{aligned} \mathcal{L}_{\text{QED}} = & Z_\psi \bar{\psi} (i\not{\partial} - Z_m m) \psi + Z_\psi g \bar{\psi} \not{A} \psi \\ & + Z_A \frac{1}{2} A^\mu (\square g_{\mu\nu} - \partial_\mu \partial_\nu) A^\nu + \frac{\lambda}{2} A^\mu \partial_\mu \partial_\nu A^\nu. \end{aligned} \quad (11.4)$$

The price we have to pay is that the renormalization constants now also enter in the Feynman rules:

$$S_0(p) = \frac{i}{Z_\psi} \frac{\not{p} + m_B}{p^2 - m_B^2 + i\epsilon},$$

$$D_0^{\mu\nu}(q) = -\frac{i}{q^2 + i\epsilon} \left(\frac{1}{Z_A} T_q^{\mu\nu} + \frac{1}{\lambda} L_q^{\mu\nu} \right), \quad (11.5)$$

$$ig \Gamma_0^\mu(p, q) = ig Z_\psi \gamma^\mu.$$

Note that we pulled out a factor ig in defining the vertex Γ_0^μ . Our momentum routing for the vertex is such that the photon momentum is $q = p_f - p_i$ and the average fermion

⁸If $Z_\Gamma = Z_\psi Z_g Z_A^{1/2}$ denotes the prefactor that we would get in front of the coupling term $\sim \bar{\psi} \not{A} \psi$, then the first condition in Eq. (11.3) is equivalent to $Z_\Gamma = Z_\psi$. To compare with the standard notation in the literature, set $Z_\Gamma = Z_1$, $Z_\psi = Z_2$ and $Z_A = Z_3$.

momentum is $p = (p_f + p_i)/2$. Along the same lines as earlier (using the projectors $T_q^{\mu\nu} = g^{\mu\nu} - q^\mu q^\nu / q^2$ and $L_q^{\mu\nu} = q^\mu q^\nu / q^2$) we obtain the inverse tree-level propagators:

$$iS_0^{-1}(p) = Z_\psi (\not{p} - m_B), \quad i(D_0^{-1})^{\mu\nu}(q) = -q^2 (Z_A T_q^{\mu\nu} + \lambda L_q^{\mu\nu}). \quad (11.6)$$

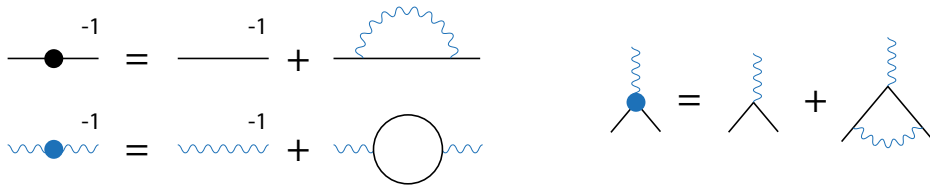
In analogy to the scalar theory, we can get the full 1PI Green functions (the inverse propagators and the fermion-photon vertex) by resumming its 1PI loop contributions. Omitting momentum arguments, this means for the fermion propagator

$$\begin{aligned} S &= S_0 + S_0 i\Sigma S_0 + S_0 i\Sigma S_0 i\Sigma S_0 + \dots \\ &= S_0 [1 + i\Sigma (S_0 + S_0 i\Sigma S_0 + \dots)] = S_0 (1 + i\Sigma S) \\ \Rightarrow S^{-1} &= S_0^{-1} - i\Sigma \quad \text{or} \quad iS^{-1} = iS_0^{-1} + \Sigma. \end{aligned} \quad (11.7)$$

$\Sigma(p)$ is the fermion self-energy, the sum of all 1PI loop contributions to the propagator. Applying the same steps to the photon propagator, we arrive at the perturbative series for the inverse propagators and the vertex:

$$\begin{aligned} iS^{-1}(p) &= iS_0^{-1}(p) + \Sigma(p), \\ i(D^{-1})^{\mu\nu}(q) &= i(D_0^{-1})^{\mu\nu}(q) + \Pi^{\mu\nu}(q), \\ \Gamma^\mu(p, q) &= \Gamma_0^\mu(p, q) + \Omega^\mu(p, q). \end{aligned} \quad (11.8)$$

The terms on the right-hand side define the fermion **self-energy** $\Sigma(p)$, the photon **vacuum polarization** $\Pi^{\mu\nu}(q)$, and the **vertex correction** $\Omega^\mu(p, q)$. To lowest order in perturbation theory they are given by the following one-loop diagrams:



Tensor decomposition. Before we proceed, let's pause for a moment and think about the general tensor decomposition of these quantities. The self-energy depends on one momentum p , so the only possible tensor structures compatible with Lorentz covariance are \not{p} and $\mathbb{1}$ (γ_5 or $\gamma_5 \not{p}$ would have the wrong sign under a parity transformation), and the coefficients can only depend on the Lorentz-invariant p^2 :

$$\Sigma(p) =: \Sigma_A(p^2) \not{p} - \Sigma_M(p^2). \quad (11.9)$$

An analogous decomposition holds for the inverse propagator itself:

$$iS^{-1}(p) = A(p^2) (\not{p} - M(p^2)), \quad (11.10)$$

which defines the **fermion mass function** $M(p^2)$, and $1/A(p^2)$ is called the fermion 'wave-function renormalization'. When substituting both equations into Eq. (11.8) we find the perturbative expansion of these dressing functions:

$$A(p^2) = Z_\psi + \Sigma_A(p^2), \quad A(p^2)M(p^2) = Z_\psi Z_m m + \Sigma_M(p^2). \quad (11.11)$$

Likewise, the only two possible tensors for the photon vacuum polarization are $g^{\mu\nu}$ and $q^\mu q^\nu$, so we can write

$$\Pi^{\mu\nu}(q) = a(q^2) g^{\mu\nu} + b(q^2) q^\mu q^\nu. \quad (11.12)$$

The scalar functions a and b cannot have poles at $q^2 = 0$ because that would correspond to an intermediate massless particle; but since the vacuum polarization is already the sum of all 1PI diagrams, intermediate propagators are excluded by definition. Now, the Ward identity $q_\mu \Pi^{\mu\nu} = 0$ entails that the vacuum polarization must be transverse to the photon momentum and therefore $a = -b q^2$. The only remaining tensor structure is then

$$\Pi^{\mu\nu}(q) = \Pi(q^2) (q^2 g^{\mu\nu} - q^\mu q^\nu) = q^2 \Pi(q^2) T_q^{\mu\nu} \quad (11.13)$$

which is proportional to the transverse projector, however with an additional factor q^2 in front.⁹ From the geometric resummation of the photon propagator analogous to Eq. (11.7) it is immediately clear that all longitudinal parts will be annihilated by $\Pi^{\mu\nu}(q)$, except for the leading tree-level term that contains the gauge parameter λ . Therefore, the longitudinal part of the photon propagator does not pick up any loop corrections beyond tree-level:

$$i(D^{-1})^{\mu\nu}(q) = -q^2 \left(\frac{T_q^{\mu\nu}}{D(q^2)} + \lambda L_q^{\mu\nu} \right), \quad D^{-1}(q^2) = Z_A - \Pi(q^2). \quad (11.14)$$

Because the longitudinal part does not get dressed, it contains no divergences and does not need to be renormalized either. This is precisely the origin of the second constraint in Eq. (11.3). As another consequence, the global factor q^2 in front of the bracket remains and, after inversion, becomes a factor $1/q^2$ in the photon propagator. Hence the photon remains massless, even with interactions, due to gauge invariance!

After inverting the above formulas, the general expressions for the fully dressed propagators and the dressed vertex become

$$\text{---}\bullet\text{---} \quad S(p) = \frac{i}{A(p^2)} \frac{\not{p} + M(p^2)}{p^2 - M^2(p^2) + i\epsilon}, \quad (11.15)$$

$$\text{---}\bullet\text{---} \quad D^{\mu\nu}(q) = -\frac{i}{q^2 + i\epsilon} \left(D(q^2) T_q^{\mu\nu} + \frac{1}{\lambda} L_q^{\mu\nu} \right), \quad (11.16)$$

$$\text{---}\bullet\text{---} \quad ig \Gamma^\mu(p, q) = ig (f_1(p^2, q^2, p \cdot q) \gamma^\mu + \dots). \quad (11.17)$$

The fermion-photon vertex is more complicated because it depends on two momenta, which leads to 12 possible tensors (we will return to this point later). In any case, when we write the vertex correction as $\Omega^\mu(p, q) = V_1(p^2, q^2, p \cdot q) \gamma^\mu + \dots$, where the dots refer to the remaining tensor structures, the general form of the vertex dressing of γ^μ is:

$$f_1(p^2, q^2, p \cdot q) = Z_\psi + V_1(p^2, q^2, p \cdot q). \quad (11.18)$$

⁹Had we solved for $b = -a/q^2$ instead, b would pick up a pole at $q^2 = 0$ contrary to what we just observed.

Renormalization conditions. The next step is to impose the renormalization conditions that are necessary to eliminate the three renormalization constants. We demand that the fermion and photon propagators become free propagators at the respective pole location, which entails

$$A(p^2 = m^2) \stackrel{!}{=} 1, \quad M(p^2 = m^2) \stackrel{!}{=} m, \quad D(q^2 = 0) \stackrel{!}{=} 1. \quad (11.19)$$

This determines the renormalization constants via Eqs. (11.11) and (11.14):

$$Z_\psi = 1 - \Sigma_A(m^2), \quad Z_\psi m_B = m - \Sigma_M(m^2), \quad Z_A = 1 + \Pi(0). \quad (11.20)$$

The resulting dressing functions, which are now finite, become

$$\begin{aligned} A(p^2) &= 1 + \Sigma_A(p^2) - \Sigma_A(m^2), \\ A(p^2)M(p^2) &= m + \Sigma_M(p^2) - \Sigma_M(m^2), \\ D^{-1}(q^2) &= 1 - \Pi(q^2) + \Pi(0). \end{aligned} \quad (11.21)$$

We could impose another condition on the vertex,

$$f_1(m^2, 0, 0) \stackrel{!}{=} 1 \quad \Rightarrow \quad Z_\psi = 1 - V_1(m^2, 0, 0), \quad (11.22)$$

but this is not necessary because it is already guaranteed by the Ward identity which allowed us to relate Z_g with Z_A . We will later see that $V_1(m^2, 0, 0) = \Sigma_A(m^2)$ is automatically satisfied in the one-loop calculation. More generally, we will also see this directly from the nonperturbative form of the vertex that follows from gauge invariance.

In summary we arrive at analogous conclusions as for the scalar theory: we can eliminate the UV divergences from the theory by imposing three renormalization conditions. We chose an onshell renormalization to make a direct connection with experiment, but our choice of renormalization conditions is arbitrary. In turn, the renormalized mass m and the renormalized charge $g = e$ are no longer predictions of the theory but they must be taken from experiment.

Fermion self-energy. As a concrete example, let us work out the one-loop contribution to the fermion self-energy in Fig. 11.1. It has the form

$$i\Sigma(p) = \int \frac{d^4k}{(2\pi)^4} (ig\gamma_\mu) S_0(k) (ig\gamma_\nu) D_0^{\mu\nu}(p-k). \quad (11.23)$$

We can ignore all renormalization constants that enter through the Feynman rules in Eq. (11.5) because they do not contribute at one-loop; the same is true for the mass renormalization so we can simply set $m_B = m$. In Feynman gauge the photon propagator is proportional to $g^{\mu\nu}$ and therefore the integral becomes

$$i\Sigma(p) = -g^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu (\not{k} + m) \gamma_\mu}{[k^2 - m^2 + i\epsilon][(p-k)^2 + i\epsilon]}. \quad (11.24)$$

Here we can exploit the formula (8.13) that we derived in the scalar theory after employing Feynman parameters and performing a Wick rotation:

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{\prod_{i=1}^2 [(k+p_i)^2 - m_i^2 + i\epsilon]} = i \int_0^1 dx I_2^{(d)}, \quad (11.25)$$

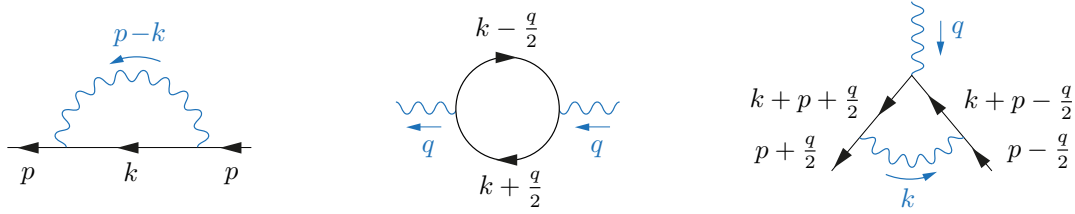


FIGURE 11.1: One-loop contributions to the fermion self-energy, the vacuum polarization and the vertex correction.

with $I_2^{(d)}$ defined in Eq. (8.20) and the remaining quantities in Eqs. (8.6–8.7). Since we want to carry on with dimensional regularization we already wrote the formula in d spacetime dimensions. In our present example we have $p_1 = -p$ and $p_2 = 0$, $m_1 = 0$ and $m_2 = m$, and $x_1 = x$, $x_2 = 1 - x$ and therefore

$$\Delta = (1 - x)(m^2 - xp^2), \quad k^\mu = l^\mu + xp^\mu. \quad (11.26)$$

Thus, the self-energy becomes

$$i\Sigma(p) = -ig^2 \int_0^1 dx \int \frac{d^d l_E}{(2\pi)^d} \frac{\gamma^\mu (\not{k} + m) \gamma_\mu}{(l_E^2 + \Delta)^2} \Big|_{k \rightarrow l + xp}. \quad (11.27)$$

We still have to work on the numerator. In d dimensions $\delta^\mu_\mu = d$, and the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ entails $\gamma^\mu \gamma_\mu = \delta^\mu_\mu = d$. This leads to $\gamma^\mu \not{k} \gamma_\mu = (2 - d) \not{k}$ and finally

$$\gamma^\mu (\not{k} + m) \gamma_\mu = (2 - d) \not{k} + md = (2 - d)(\not{l} + xp) + md. \quad (11.28)$$

Factors of l^μ in the numerator are easily manageable because

$$\begin{aligned} \int \frac{d^d l_E}{(2\pi)^d} \frac{l^\mu}{(l_E^2 + \Delta)^2} &= 0, \\ \int \frac{d^d l_E}{(2\pi)^d} \frac{l^\mu l^\nu}{(l_E^2 + \Delta)^2} &= -\frac{1}{d} g^{\mu\nu} \int \frac{d^d l_E}{(2\pi)^d} \frac{l_E^2}{(l_E^2 + \Delta)^2}. \end{aligned} \quad (11.29)$$

The first integral vanishes due to symmetry (replace $l \rightarrow -l$ in the integrand), and so does the second for $\mu \neq \nu$. For $\mu = \nu$ it must be proportional to $g^{\mu\nu}$ by Lorentz invariance, and by contracting the indices one verifies that the prefactor on the r.h.s. is correct. Hence, Eq. (11.28) becomes

$$\gamma^\mu (\not{k} + m) \gamma_\mu = (2 - d) xp + md \quad \Rightarrow \quad i\Sigma(p) = -ig^2 \int_0^1 dx [(2 - d) xp + md] I_2^{(d)}.$$

By comparing with Eq. (11.9) we read off the self-energy contributions:

$$\Sigma_A(p^2) = g^2(d - 2) \int dx x I_2^{(d)}, \quad \Sigma_M(p^2) = g^2 md \int dx I_2^{(d)}. \quad (11.30)$$

Setting now $d = 4 - \epsilon$, taking the limit $\epsilon \rightarrow 0$, and inserting the result (8.29) for $I_2^{(d)}$ in dimensional regularization, we arrive at

$$\begin{aligned}\Sigma_A(p^2) &= \frac{\alpha}{2\pi} \int dx x \left[\frac{2}{\epsilon} - \gamma + \ln \frac{4\pi M^2}{\Delta} - 1 \right], \\ \Sigma_M(p^2) &= \frac{\alpha m}{\pi} \int dx \left[\frac{2}{\epsilon} - \gamma + \ln \frac{4\pi M^2}{\Delta} - \frac{1}{2} \right],\end{aligned}\tag{11.31}$$

where we have also replaced the coupling by $g^2 = 4\pi\alpha$. This is the one-loop fermion self-energy in dimensional regularization. We see that the method is completely analogous to the scalar theory; from Eq. (11.30) we could have equally derived the result in Pauli-Villars regularization. In both cases the expressions contain divergent and finite pieces: in dimensional regularization the divergences are of the form $\sim 1/\epsilon$ whereas with PV regularization they are logarithmic.

To arrive at finite expressions, we apply the renormalization procedure outlined above. That is, we subtract the self-energy at $p^2 = m^2$:

$$\begin{aligned}\Sigma_A(p^2) - \Sigma_A(m^2) &= \frac{\alpha}{2\pi} \int dx x \ln \frac{\Delta_m}{\Delta}, & \frac{\Delta_m}{\Delta} &= \frac{m^2(1-x)}{m^2 - xp^2}, \\ \Sigma_M(p^2) - \Sigma_M(m^2) &= \frac{\alpha m}{\pi} \int dx \ln \frac{\Delta_m}{\Delta},\end{aligned}\tag{11.32}$$

which makes the dressing functions in Eq. (11.21) finite. Note that the logarithm develops a branch cut for negative arguments. Since $0 < x < 1$, the condition is

$$\frac{p^2}{m^2} > \frac{1}{x} > 1,\tag{11.33}$$

and therefore the branch cut starts at $p^2 = m^2$. This is just what we anticipated with the Källén-Lehmann representation, cf. Fig. 6.2. Due to the self-energy correction the fermion can split into a fermion plus a photon (and, when going to higher orders in perturbation theory, arbitrarily many photons), but since the photon is massless, the multiparticle continuum that produces the cut starts at $p^2 = (m + m_\gamma)^2 = m^2$.

From Eq. (11.21) we extract the one-loop result for the mass function $M(p^2)$:

$$\begin{aligned}M(p^2) &= \frac{m + \Sigma_M(p^2) - \Sigma_M(m^2)}{1 + \Sigma_A(p^2) - \Sigma_A(m^2)} \\ &\approx m + \Sigma_M(p^2) - \Sigma_M(m^2) - m (\Sigma_A(p^2) - \Sigma_A(m^2)) \\ &= m \left[1 + \frac{\alpha}{\pi} \int dx \left(1 - \frac{x}{2} \right) \ln \frac{\Delta_m}{\Delta} \right],\end{aligned}\tag{11.34}$$

which inherits the branch cut for $p^2 > m^2$. It is also instructive to work out the explicit form for large spacelike $Q^2 := -p^2 \gg m^2$. In that case

$$\ln \frac{\Delta_m}{\Delta} \approx \ln \frac{m^2(1-x)}{x Q^2} \approx -\ln \frac{Q^2}{m^2} + \dots\tag{11.35}$$

and therefore the mass function falls off logarithmically with Q^2 (see Fig. 11.2):

$$M(p^2) = m \left[1 - \frac{\alpha}{\pi} \ln \frac{Q^2}{m^2} \int dx \left(1 - \frac{x}{2} \right) + \dots \right] = m \left[1 - \frac{3\alpha}{4\pi} \ln \frac{Q^2}{m^2} + \dots \right].\tag{11.36}$$

Vacuum polarization. We already discussed the general properties of the vacuum polarization (transversality and analyticity at $q^2 = 0$) above. The generally allowed form for the vacuum polarization tensor is

$$\Pi^{\mu\nu}(q) = \Pi(q^2)(q^2 g^{\mu\nu} - q^\mu q^\nu) + \tilde{\Pi}(q^2) g^{\mu\nu}, \quad (11.37)$$

but the Ward identity $q_\mu \Pi^{\mu\nu} = 0$ enforces $\tilde{\Pi}(q^2) = 0$. This is indeed true at each order in perturbation theory provided that the regularization method respects gauge invariance.

One can treat the one-loop expression in Fig. 11.1 in complete analogy to the fermion self-energy example. The one-loop Feynman graph has the form

$$\begin{aligned} i\Pi^{\mu\nu}(q) &= -\text{Tr} \int \frac{d^4 k}{(2\pi)^4} (ig\gamma^\mu) S_0(k_+) (ig\gamma^\nu) S_0(k_-) \\ &= -g^2 \text{Tr} \int \frac{d^4 k}{(2\pi)^4} \frac{\gamma^\mu (\not{k}_+ + m) \gamma^\nu (\not{k}_- + m)}{[k_+^2 - m^2 + i\epsilon][k_-^2 - m^2 + i\epsilon]}. \end{aligned} \quad (11.38)$$

(Ex)

The calculation is a bit lengthier but still manageable; the result is

$$\begin{aligned} \Pi(q^2) &= -8g^2 \int dx x(1-x) I_2^{(d)}, \\ \tilde{\Pi}(q^2) &= -4g^2 \int dx \left(I_2^{(d)} \Delta + \left(1 - \frac{2}{d}\right) \tilde{I}_2^{(d)} \right) \end{aligned} \quad (11.39)$$

with $\Delta = m^2 - x(1-x)q^2$. The integrals are given in Eq. (8.32), and with their explicit form it is easy to check that $\tilde{\Pi}(q^2)$ vanishes indeed in dimensional regularization. However, this is not true for a momentum cutoff: in that case $\tilde{\Pi}(q^2)$ is not only nonzero but also develops a quadratic divergence (as one would infer from a dimensional analysis of the diagram) due to the appearance of $\tilde{I}_2^{(d)}$. Hence, a cutoff regulator breaks gauge invariance, and therefore it is not the optimal choice when dealing with gauge theories (unless one knows how to eliminate the contamination from such ‘gauge parts’).

The transverse piece, on the other hand, is only logarithmically divergent. In dimensional regularization it is given by

$$\Pi(q^2) = -\frac{2\alpha}{\pi} \int dx x(1-x) \left(\frac{2}{\epsilon} - \gamma + \ln \frac{4\pi M^2}{\Delta} \right). \quad (11.40)$$

After performing the subtraction in Eq. (11.21) it becomes

$$\Pi(q^2) - \Pi(0) = -\frac{2\alpha}{\pi} \int dx x(1-x) \ln \frac{\Delta_0}{\Delta}, \quad \frac{\Delta_0}{\Delta} = \frac{m^2}{m^2 - x(1-x)q^2}. \quad (11.41)$$

Notice again the branch cut from the logarithm: since $0 < x(1-x) < \frac{1}{4}$ the condition is now

$$\frac{q^2}{m^2} > \frac{1}{x(1-x)} > 4 \quad \Rightarrow \quad q^2 > 4m^2 \quad (11.42)$$

as it should be, because $2m$ is the threshold for e^+e^- pair creation.

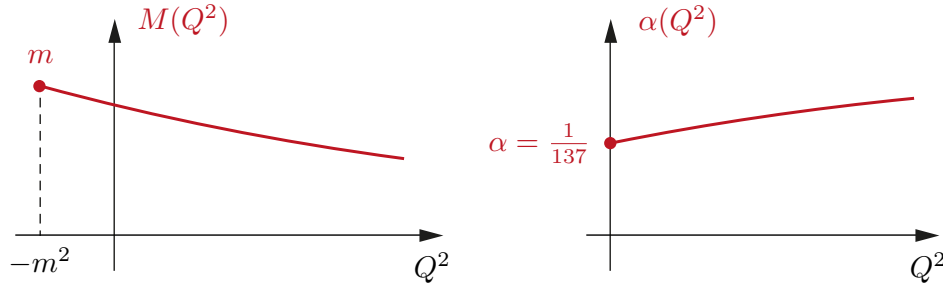


FIGURE 11.2: One-loop behavior of the fermion mass function and running coupling.

Running coupling. The vacuum polarization has another practical relevance. We can define an effective **running coupling** as the product of the coupling constant α and the photon dressing:

$$\alpha(q^2) := \alpha D(q^2) = \frac{\alpha}{1 - \Pi(q^2) + \Pi(0)}. \quad (11.43)$$

It is fully determined by the vacuum polarization, so for positive $q^2 > 4m^2$ it inherits the branch cut from Eq. (11.41). To obtain the form for large spacelike $Q^2 := -q^2 \gg m^2$ we plug in the one-loop result:

$$\frac{\Delta_0}{\Delta} \approx \frac{m^2}{x(1-x)Q^2} \Rightarrow \Pi(q^2) - \Pi(0) = \frac{2\alpha}{\pi} \left[\ln \frac{Q^2}{m^2} \underbrace{\int dx x(1-x)}_{1/6} + \dots \right],$$

and we find that the running coupling rises logarithmically with Q^2 as in Fig. 11.2:

$$\alpha(q^2) \approx \frac{\alpha}{1 - \frac{\alpha}{3\pi} \ln \frac{Q^2}{m^2}}. \quad (11.44)$$

The reason behind the definition (11.43) is the following. Suppose we reformulate our renormalization conditions (11.19) in terms of $A(p^2)$, $M(p^2)$ and $\alpha(q^2)$:

$$A(p^2 = m^2) \stackrel{!}{=} 1, \quad M(p^2 = m^2) \stackrel{!}{=} m, \quad \alpha(q^2 = 0) \stackrel{!}{=} \alpha, \quad (11.45)$$

which makes the nature of m and α as an external input to QED explicit. With this we can calculate the momentum dependence of $M(p^2)$ and $\alpha(q^2)$, for example in one-loop perturbation theory as in Fig. 11.2. However, these curves would look the *same* if we had not renormalized at $p^2 = m^2$ and $q^2 = 0$ but at some arbitrary scales $p^2 = \mu^2$ and $q^2 = \nu^2$, provided that we used $M(\mu^2)$ and $\alpha(\nu^2)$ as the new input values. Such a change cannot affect $M(p^2)$, $\alpha(q^2)$ nor any other prediction of the theory. On the other hand, this only holds as long as $M(p^2)$ and $\alpha(q^2)$ do not *additionally* depend on the renormalization point (that is, they must be *renormalization-group invariant*).

This is true for the fermion mass function, which we can easily confirm. From the relations (11.2) between the bare and renormalized fields one immediately derives the relations between the bare and renormalized n-point functions, for example

$$\langle \Omega | \text{T} \psi(x)_B \bar{\psi}_B(y) | \Omega \rangle = Z_\psi \langle \Omega | \text{T} \psi(x) \bar{\psi}(y) | \Omega \rangle \quad (11.46)$$

and therefore $S_B(p) = Z_\psi S(p)$. When we impose the renormalization conditions at some renormalization point $p^2 = \mu^2$, Z_ψ will depend on the renormalization point and so will all the Green functions of

the theory. For example, the renormalized propagator is $S(p, \mu)$ and its dressing functions $A(p^2)$ and $M(p^2)$ have the form

$$A_B(p^2) = \frac{1}{Z_\psi(\mu^2)} A(p^2, \mu^2) \quad \text{but} \quad M_B(p^2) = M(p^2). \quad (11.47)$$

Due to our definition (11.10) the dependence on the renormalization constant Z_ψ is entirely carried by the function $A(p^2)$, whereas $M(p^2)$ stays unrenormalized: $M_B(p^2) = M(p^2)$. The divergences must therefore cancel in the mass function even if we had not renormalized the theory. Because there is no Z factor that relates the bare with the renormalized mass function, $M(p^2)$ also cannot depend on μ and its interpretation as a ‘running fermion mass’ is acceptable.

The analogous combination for the coupling α must be of the form

$$(g^2 f(q^2))_B = g^2 f(q^2). \quad (11.48)$$

The relation $Z_g Z_A^{1/2}$ in Eq. (11.3) that follows from the Ward identity suggests to identify $f(q^2)$ with the photon dressing function, because also

$$(g^2 D(q^2))_B = Z_g^2 Z_A (g^2 D(q^2)) = g^2 D(q^2) \quad (11.49)$$

stays unrenormalized, i.e., it is a renormalization-group invariant.

Since the values for m and α are an input to QED, the theory ‘knows’ how the electron mass and its charge evolve with the momentum scale, and this scale dependence is encoded in the functional form of $M(p^2)$ and $\alpha(q^2)$. With $e^2 = 4\pi\alpha$ we may interpret $\alpha(q^2)$ as the effective momentum dependence of the electron charge. Nonrelativistically, the spacelike Q^2 dependence translates into a potential between two electrons (or an electron and a positron). If we pull two electrons infinitely far apart, we probe the coupling at $Q^2 = 0$; this is where we extract $\alpha(0) \approx \frac{1}{137}$ experimentally. The rise of $\alpha(q^2)$ at $Q^2 > 0$ can be viewed as a **screening of the charge**: at $Q^2 = 0$, the electron is screened by a cloud of virtual e^+e^- pairs, but at higher Q^2 (smaller distances) we eventually penetrate this charge cloud and see more of the electron’s ‘true’ charge which is larger. Hence the name ‘vacuum polarization’, because the vacuum behaves like a polarizable medium.

On the other hand, the rise of $\alpha(q^2)$ happens extremely slowly and the coupling remains the same over many orders of magnitude. Between $Q^2 = 0 \dots 30 \text{ GeV}^2$, this rise is only about 1% from the e^+e^- loop and $\sim 5\%$ in total (including heavier leptons and also quarks). This is good news because $\alpha(q^2)$ is also the expansion parameter in perturbation theory. The result in Fig. 11.2 was obtained at one-loop; if $\alpha(q^2)$ would rise dramatically with the momentum, we could forget about applying perturbation theory at larger Q^2 . Fortunately the coupling is still small at large momenta, so the perturbative treatment is justified.

Nevertheless, the fact that this rise continues indefinitely casts doubt on the behavior of the theory at very small distances or very large energies; it is referred to as the **Landau pole** of QED. The one-loop formula develops a pole at $Q^2 \sim (10^{277} \text{ GeV})^2$, which is completely irrelevant in practice because electromagnetism eventually merges with the weak interactions and even the Planck scale 10^{19} GeV is much lower. Still, this implies that QED by itself is not a well-defined theory at high energies. The situation in QCD is reversed: $\alpha(Q^2)$ falls off with higher momenta due to asymptotic freedom, so the theory is well-defined in the ultraviolet.

Fermion-photon vertex. Before discussing the perturbative one-loop result for the vertex correction in Fig. 11.1, let's have a look at the general properties of the fermion-photon vertex. We use the same kinematics as earlier: $q = p_+ - p_-$ is the incoming photon momentum and $p = (p_+ + p_-)/2$ is the average momentum of the fermions. The squared fermion momenta are given by

$$p_{\pm}^2 = p^2 + \frac{q^2}{4} \pm p \cdot q \quad \Rightarrow \quad p_+^2 - p_-^2 = 2p \cdot q, \quad (11.50)$$

so the onshell limit $p_{\pm}^2 = m^2$ corresponds to $p \cdot q = 0$ and $p^2 = m^2 - q^2/4$. The dependence of the vertex on two independent momenta leads to 12 possible tensors:

$$\{\gamma^\mu, p^\mu, q^\mu\} \times \{\mathbb{1}, \not{p}, \not{q}, [\not{p}, \not{q}]\}. \quad (11.51)$$

This looks rather hopeless, but fortunately gauge invariance provides us with some ordering principle. We mentioned that local $U(1)$ gauge invariance is equivalent to the statement that the photons couple to fermions through the conserved vector current of the global $U(1)$ symmetry. A current that is classically conserved induces **Ward-Takahashi identities (WTIs)** for the Green functions of the theory. These are identities that relate an n -point function to $(n-1)$ -point functions. Without proof, we state the WTI for the fermion-photon vertex:

$$q_\mu \Gamma^\mu(p, q) = iS^{-1}(p + \frac{q}{2}) - iS^{-1}(p - \frac{q}{2}), \quad (11.52)$$

which holds not only for $q^\mu \rightarrow 0$ but in general, and it tells us that the vertex is partially determined by the inverse fermion propagator.

The WTI has several practical consequences. First, we can work it out explicitly using the tensor decomposition (11.10) for the inverse fermion propagator. Abbreviating $B(p^2) = A(p^2)M(p^2)$, as well as $A(p_{\pm}^2) = A_{\pm}$ and $B(p_{\pm}^2) = B_{\pm}$, it takes the form

$$\begin{aligned} q_\mu \Gamma^\mu(p, q) &= \left(\not{p} + \frac{\not{q}}{2}\right) A_+ - \left(\not{p} - \frac{\not{q}}{2}\right) A_- - B_+ + B_- \\ &= \underbrace{\frac{A_+ + A_-}{2}}_{=: \bar{A}} \not{q} + \underbrace{\frac{A_+ - A_-}{2p \cdot q}}_{=: \Delta_A} 2p \cdot q \not{p} - \underbrace{\frac{B_+ - B_-}{2p \cdot q}}_{=: \Delta_B} 2p \cdot q \\ &= q_\mu \underbrace{\left[\bar{A} \gamma^\mu + 2p^\mu (\Delta_A \not{p} - \Delta_B)\right]}_{=: \Gamma_{\text{BC}}^\mu(p, q)}. \end{aligned} \quad (11.53)$$

The quantities Δ_A and Δ_B are difference quotients because $2p \cdot q = p_+^2 - p_-^2$, and in the limit $p_+^2 = p_-^2$ they become the derivatives of the dressing functions $A(p^2)$ and $B(p^2)$ with respect to p^2 . The bracket in the last line defines the **Ball-Chiu vertex** which is the part of the vertex that is constrained by gauge invariance. Consequently, the full vertex can only differ by a purely transverse part that does not contribute to the WTI:

$$\Gamma^\mu(p, q) = \Gamma_{\text{BC}}^\mu(p, q) + \Gamma_{\text{T}}^\mu(p, q). \quad (11.54)$$

The transverse part cannot have analytic poles at $q^2 = 0$ because that would again contradict its 1PI property. In combination with the transversality condition, one can

show that this has the consequence that the transverse part must be at least linear in q^μ , so it vanishes for $q^\mu \rightarrow 0$. It depends on eight remaining tensors, and in analogy to the vacuum polarization one can construct appropriate tensor bases so that its dressing functions are free of kinematic singularities and constraints at $q^2 = 0$.

We are now ready to verify the first relation in Eq. (11.3). It follows from the fact that the WTI holds for renormalized and unrenormalized quantities alike: if we define $Z_\Gamma = Z_\psi Z_g Z_A^{1/2}$, then the argument that we used in Eq. (11.46) entails

$$\Gamma_B = \frac{1}{Z_\Gamma} \Gamma, \quad S_B^{-1} = \frac{1}{Z_\psi} S^{-1}. \quad (11.55)$$

Therefore, the WTI for the bare vertex and propagator is identical to the renormalized WTI in Eq. (11.52), except that an additional factor $1/Z_\Gamma$ appears on the left-hand side and $1/Z_\psi$ on the right. This, in turn, requires $Z_\Gamma = Z_\psi$ and consequently $Z_g Z_A^{1/2} = 1$.

Since that identity eliminates the renormalization constant Z_g we had no freedom anymore to renormalize the vertex. Instead, we claimed that $f(m^2, 0, 0) = 1$ will be automatically ensured by the WTI. Now we can see how this comes about: we renormalized the fermion propagator at $p^2 = m^2$ and the photon propagator at $q^2 = 0$; the corresponding onshell limit for the vertex is

$$\Gamma^\mu(p, q) \rightarrow A(m^2) \gamma^\mu + 2p^\mu (A'(m^2) \not{p} - B'(m^2)). \quad (11.56)$$

This is the *exact* form of the vertex in that limit because the transverse part does not contribute. With our renormalization condition $A(m^2) = 1$ the dressing function of the γ^μ component is indeed $f(m^2, 0, 0) = A(m^2) = 1$, as advertised.

Electromagnetic form factors. In onshell scattering matrix elements we additionally need to attach Dirac spinors to the vertex, so we must work out the quantity $\bar{u}(p_+) \Gamma^\mu(p, q) u(p_-)$. In onshell kinematics $p \cdot q = 0$ and $p^2 = m^2 - q^2/4$, so q^2 is the only remaining Lorentz-invariant. In that case the WTI (11.52) reduces to the **Ward identity**

$$q_\mu \bar{u}(p_+) \Gamma^\mu(p, q) u(p_-) = 0. \quad (11.57)$$

It follows immediately from taking (11.56) in the onshell limit $p \cdot q = 0$ and exploiting the Dirac equation for the onshell spinors:

$$\begin{aligned} \bar{u}(p_+) \not{p}_+ &= m \bar{u}(p_+) & \Rightarrow & \quad \bar{u}(p_+) \not{q} u(p_-) = \bar{u}(p_+) (\not{p}_+ - \not{p}_-) u(p_-) = 0. \\ \not{p}_- u(p_-) &= m u(p_-) \end{aligned} \quad (11.58)$$

On the other hand, starting from the tensor structures in Eq. (11.51) we can write down the most general onshell decomposition of the current. By judicious use of the Dirac equations one can eliminate all slashed quantities, for example

$$\bar{u}(p_+) \not{p} u(p_-) = \bar{u}(p_+) \frac{\not{p}_+ + \not{p}_-}{2} u(p_-) = m u(p_-). \quad (11.59)$$

This leaves three possible dressing functions which can only depend on q^2 :

$$\bar{u}(p_+) \Gamma^\mu(p, q) u(p_-) = \bar{u}(p_+) [a \gamma^\mu + b p^\mu + c q^\mu] u(p_-). \quad (11.60)$$

The Ward identity then enforces $c = 0$, so we are left with γ^μ and p^μ . Using the Dirac equations it is easy to prove the **Gordon identity**

(Ex)

$$\bar{u}(p_+) \left[\gamma^\mu - \frac{p^\mu}{m} - \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right] u(p_-), \quad (11.61)$$

with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, which allows us to eliminate p^μ in favor of $\sigma^{\mu\nu} q_\nu$. The final result for the onshell vertex is

$$\bar{u}(p_+) \Gamma^\mu(p, q) u(p_-) = \bar{u}(p_+) \left[F_1(q^2) \gamma^\mu + F_2(q^2) \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right] u(p_-). \quad (11.62)$$

$F_1(q^2)$ and $F_2(q^2)$ are called the **electromagnetic Dirac and Pauli form factors**, respectively, and $F_2(0)$ is the anomalous magnetic moment of the fermion.

Along the same lines we can also apply spinors to Eq. (11.56), which amounts to replacing $\not{p} \rightarrow m$ and eliminating p^μ using the Gordon identity. The form factors become

$$F_1(q^2) = A(m^2) - C(m^2) + q^2 [\dots], \quad F_2(q^2) = C(m^2) + [\dots], \quad (11.63)$$

where $C(m^2) := -2m(mA'(m^2) - B'(m^2)) = 2mA(m^2)M'(m^2)$ and the dots refer to further contributions coming from the transverse part of the vertex. Observe that the renormalization condition $A(m^2) = 1$ does *not* lead to $F_1(0) = 1$; we have to choose $A(m^2) = 1 + C(m^2)$ instead. This is equivalent to the following modification of the renormalization conditions in Eq. (11.19):

$$iS^{-1}(\not{p}) \Big|_{\not{p} \rightarrow m} \stackrel{!}{=} 0 \Rightarrow M(m^2) = m, \quad \frac{d}{d\not{p}} iS^{-1}(\not{p}) \Big|_{\not{p} \rightarrow m} \stackrel{!}{=} 1 \Rightarrow A(m^2) = 1 + C(m^2). \quad (11.64)$$

Here we view the propagator as a function of \not{p} , with $\not{p}^2 = p^2$, and thereby also take the derivative of the dressing functions. This will also modify the renormalization constants in Eq. (11.20),

$$Z_\psi = 1 + C(m^2) - \Sigma_A(m^2), \quad Z_\psi m_B = (1 + C(m^2)) m - \Sigma_M(m^2), \quad (11.65)$$

as well as the result for the renormalized dressing functions:

$$\begin{aligned} A(p^2) &= 1 + C(m^2) + \Sigma_A(p^2) - \Sigma_A(m^2), \\ A(p^2)M(p^2) &= (1 + C(m^2)) m + \Sigma_M(p^2) - \Sigma_M(m^2). \end{aligned} \quad (11.66)$$

If we write $F_1(q^2) = Z_\psi + \delta F_1(q^2)$, then the Ward identity gives the result $\delta F_1(0) = \Sigma_A(m^2) - C(m^2)$.

Perturbative result for the vertex correction. The one-loop calculation for the vertex correction in Fig. 11.1 is considerably more complicated than the self-energy calculation but otherwise completely analogous. Starting from Eq. (11.8), the diagram is given by

$$\begin{aligned} ig \Omega^\mu(p, q) &= \bar{u}(p_+) \int \frac{d^4 k}{(2\pi)^4} (ig \gamma_\rho) S(k_+) (ig \gamma_\mu) S(k_-) (ig \gamma_\sigma) D^{\rho\sigma}(k) u(p_-) \\ &= g^3 \int \frac{d^4 k}{(2\pi)^4} \frac{\bar{u}(p_+) \gamma_\rho (\not{k}_+ + m) \gamma_\mu (\not{k}_- + m) \gamma^\rho u(p_-)}{[k_+^2 - m^2 + i\epsilon][k_-^2 - m^2 + i\epsilon][k^2 + i\epsilon]}. \end{aligned} \quad (11.67)$$

Inserting the formula (8.13) with $p_1 = p + \frac{q}{2}$, $p_2 = p - \frac{q}{2}$, $p_3 = 0$, $m_1 = m_2 = m$ and $m_3 = 0$ yields

$$\Omega^\mu(p, q) = -2g^2 \int \underbrace{dx dy dz \delta(x + y + z - 1)}_{:= d\omega} \int \frac{d^4 l_E}{(2\pi)^4} \frac{\mathcal{N}}{(l_E^2 + \Delta)^3}, \quad (11.68)$$

where \mathcal{N} is the numerator with the replacement $k^\mu \rightarrow l^\mu - a p^\mu - \frac{b}{2} q^\mu$ and the function in the denominator is given by

$$\Delta = a^2 m^2 + (b^2 - a^2) \frac{q^2}{4}, \quad a = x + y = 1 - z, \quad b = x - y. \quad (11.69)$$

The hardest part is working out the numerator. After some pages of calculation, the vertex correction becomes

(Ex)

$$\Omega^\mu(p, q) = 2g^2 \int d\omega \bar{u}(p_+) \left[H_1 \gamma^\mu + H_2 \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right] u(p_-), \quad (11.70)$$

where

$$\begin{aligned} H_1 &= \frac{(d-2)^2}{d} \tilde{I}_3^{(d)} + 2(2m^2(a(1+a) - 1) - \Delta + (1-a)q^2) I_3^{(4)}, \\ H_2 &= 4m^2 a(1-a) I_3^{(4)}. \end{aligned} \quad (11.71)$$

The divergent parts can only come from factors $\sim l_E^2$ in the numerator which lead to the divergent integral $\tilde{I}_3^{(d)}$. All other contributions are finite and proportional to $I_3^{(4)}$, so we already took the limit $d \rightarrow 4$ for those. Note in particular that H_2 is finite, i.e., the Pauli form factor is free of divergences.

Using dimensional regularization for $\tilde{I}_3^{(d)}$, the form factors become

$$\begin{aligned} F_1(q^2) &= Z_\psi + \frac{\alpha}{2\pi} \int d\omega \left[\frac{2}{\epsilon} - \gamma + \ln \frac{4\pi M^2}{\Delta} - 3 + \frac{2m^2(a(1+a) - 1) + (1-a)q^2}{\Delta} \right], \\ F_2(q^2) &= m^2 \frac{\alpha}{\pi} \int d\omega \frac{a(1-a)}{\Delta}. \end{aligned} \quad (11.72)$$

At $q^2 = 0$, we have $\Delta = a^2 m^2$, and with $a = 1 - z$ from Eq. (11.69) we find

$$F_2(0) = \frac{\alpha}{\pi} \int d\omega \frac{z}{1-z} = \frac{\alpha}{\pi} \int_0^1 dz \int_0^{1-z} dy \frac{z}{1-z} = \frac{\alpha}{\pi} \int_0^1 dz z = \frac{\alpha}{2\pi}. \quad (11.73)$$

This is Schwinger's famous result for the **anomalous magnetic moment of the electron** at one-loop order. Inserting $\alpha \approx \frac{1}{137}$, the numerical value is about 1‰: $F_2(0) = 0.0011614$, plus higher orders in perturbation theory. Compare this with the experimental result: $F_2(0)_{\text{exp}} = 0.0011597$.

Another check is whether the Ward identity truly holds. From Eq. (11.72) we infer

$$F_1(0) = Z_\psi + \frac{\alpha}{2\pi} \left[\frac{1}{2} \left(\frac{2}{\epsilon} - \gamma + \ln \frac{4\pi M^2}{m^2} - 1 \right) - \int da \left(\frac{2(1-a)}{a} + a \ln a^2 \right) \right]. \quad (11.74)$$

On the other hand, with $C(m^2)$ defined in Eq. (11.63), the fermion renormalization constant obtained from Eq. (11.31) is

$$\begin{aligned} Z_\psi &= 1 + C(m^2) - \Sigma_A(m^2) \\ &= 1 + \frac{\alpha}{2\pi} \left[-\frac{1}{2} \left(\frac{2}{\epsilon} - \gamma + \ln \frac{4\pi M^2}{m^2} - 1 \right) + \int da (1-a) \left(\frac{2(1+a)}{a} + \ln a^2 \right) \right], \end{aligned} \quad (11.75)$$

and so we have in total

$$F_1(0) = 1 + \frac{\alpha}{2\pi} \int da [2(1-a) + (1-2a) \ln a^2] = 1. \quad (11.76)$$

12 Path integrals

[Unfinished]

Path integrals in quantum mechanics. We want to compute the transition matrix element

$$A = \langle x_f, t_f | x_i, t_i \rangle = \langle x_f | e^{-iH(t_f-t_i)} | x_i \rangle. \quad (12.1)$$

To do so, we split the difference $t_f - t_i$ into N equal time steps:

$$\{x_f, t_f\} = \{x_N, t_N\}, \quad \{x_i, t_i\} = \{x_0, t_0\}, \quad t_f - t_i = N \delta t, \quad (12.2)$$

where we will take the limit $N \rightarrow \infty$ and $\delta t \rightarrow 0$ in the end. Next, we insert the completeness relation (??) for each time slice from $\{x_1, t_1\}$ to $\{x_{N-1}, t_{N-1}\}$:

$$A = \int dx_1 \dots dx_{N-1} \prod_{k=0}^{N-1} \underbrace{\langle x_{k+1}, t_{k+1} | x_k, t_k \rangle}_{=: A_k}. \quad (12.3)$$

To evaluate A_k we go back into the Schrödinger picture:

$$A_k = \langle x_{k+1} | \underbrace{e^{-iH(t_{k+1}-t_k)}}_{e^{-iH\delta t}} | x_k \rangle = \int dp_k \langle x_{k+1} | p_k \rangle \langle p_k | e^{-iH\delta t} | x_k \rangle. \quad (12.4)$$

Here we additionally inserted a completeness relation for the momenta. If we assume that the Hamilton operator has the form $H(P, X) = f(P) + V(X)$, then from the Baker-Campbell-Hausdorff formula

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]+\dots} \Rightarrow e^{-iH\delta t} = e^{-i\delta t f(P)} e^{-i\delta t V(X)} e^{\frac{1}{2}(\delta t)^2 [f(P), V(X)]+\dots}$$

and in the limit $\delta t \rightarrow 0$ we can neglect the term $\sim (\delta t)^2$. Since $|p_k\rangle$ and $|x_k\rangle$ are eigenstates of P and X at the time t_k , respectively, we can replace

$$\langle p_k | e^{-iH(P,X)\delta t} | x_k \rangle = \langle p_k | x_k \rangle e^{-iH(p_k, x_k)\delta t} = \frac{e^{-ip_k x_k}}{\sqrt{2\pi}} e^{-iH(p_k, x_k)\delta t} \quad (12.5)$$

to arrive at

$$A_k = \int \frac{dp_k}{2\pi} e^{i\delta t \left[p_k \frac{x_{k+1} - x_k}{\delta t} - H(p_k, x_k) \right]} \xrightarrow{N \rightarrow \infty} \int \frac{dp_k}{2\pi} e^{i\delta t [p_k \dot{x}_k - H(p_k, x_k)]}. \quad (12.6)$$

Consider now specifically a Hamiltonian of the form

$$H(P, X) = \frac{P^2}{2m} + V(X). \quad (12.7)$$

If we give the time variable a small imaginary part, $\delta t \rightarrow \delta t(1 - i\epsilon)$, the kinetic term becomes

$$A_k = \int \frac{dp_k}{2\pi} e^{i\delta t(1-i\epsilon) \left[p_k \frac{x_{k+1} - x_k}{\delta t} - \frac{p_k^2}{2m} - V(x_k) \right]}, \quad (12.8)$$

which is solved by the Gaussian integral

$$\int dp e^{-\frac{1}{2}ap^2+bp} = \sqrt{\frac{2\pi}{a}} e^{\frac{b^2}{2a}}, \quad \text{Re } a > 0, \quad (12.9)$$

namely as

$$A_k = \sqrt{\frac{m}{2i\pi\delta t}} e^{\frac{im}{2\delta t}(x_{k+1}-x_k)^2} \xrightarrow{N \rightarrow \infty} C e^{i\delta t [\frac{m}{2}\dot{x}_k^2 - V(x_k)]} = C e^{i\delta t L(x_k, \dot{x}_k)}. \quad (12.10)$$

Putting this back into Eq. (??), we find

$$A = \lim_{N \rightarrow \infty} C^N \int dx_1 \dots dx_{N-1} e^{i \sum_{k=0}^{N-1} \delta t L}. \quad (12.11)$$

This expression has a rather intuitive physical interpretation, cf. Fig. ?? . Suppose we define a function $x(t)$ such that $x(t_k) = x_k$. Then at each time t_k we integrate over all possible values of x_k , which means that we integrate over a large class of functions $x(t)$ with fixed boundary values $x(t_0) = x_0$ and $x(t_N) = x_N$. We denote this by a **path integral**:

$$A =: \int_{x(t_0)=x_0}^{x(t_N)=x_N} \mathcal{D}x e^{i \int_{t_0}^{t_N} dt L} = \int \mathcal{D}x e^{iS[x]}, \quad (12.12)$$

where $S[x]$ is the action evaluated between the times t_i and t_f . This result is known as the **Feynman-Kac formula**. Note that in order to make the expression well-defined by had to add the imaginary contribution to the time. Wick rotation. One can show that the integral is well-defined in the limit $N \rightarrow \infty$ for the specific form (??) of the Hamiltonian, which is what we will need for the generalization to quantum field theory.

We can repeat the idea for matrix elements of position space operators. The matrix element of an operator $X(t)$ at time t , with $t_f > t > t_i$, is

$$\begin{aligned} \langle x_N, t_N | X(t) | x_0, t_0 \rangle &= \int dx \langle x_N, t_N | X(t) | x, t \rangle \langle x, t | x_0, t_0 \rangle \\ &= \int dx \langle x_N, t_N | x, t \rangle x(t) \langle x, t | x_0, t_0 \rangle \\ &= \int dx \left[\int_{x(t)=x}^{x(t_N)=x_N} \mathcal{D}x e^{i \int_t^{t_N} dt L} \right] x(t) \left[\int_{x(t_0)=x_0}^{x(t)=x} \mathcal{D}x e^{i \int_{t_0}^t dt L} \right] \\ &= \int_{x(t_0)=x_0}^{x(t_N)=x_N} \mathcal{D}x e^{i \int_{t_0}^{t_N} dt L} x = \int \mathcal{D}x e^{iS[x]} x(t). \end{aligned} \quad (12.13)$$

The two path integrals in the third line run from x_0 to fixed x and then from x to x_N , but since we integrate over all x this is equivalent to the path integral from x_0 to x_N . Repeating this for a product of operators at different times will automatically produce a **time-ordering** for them, so we can write

$$\langle x_N, t_N | T X(t_1) \dots X(t_n) | x_0, t_0 \rangle = \int \mathcal{D}x e^{iS[x]} x(t_1) \dots x(t_n). \quad (12.14)$$

Finally we would like to know how the transition amplitude behaves for $t_f \rightarrow \infty$, $t_i \rightarrow -\infty$. We insert a complete set of orthonormal eigenstates of the Hamiltonian, with $H|\Omega\rangle = 0$, and denote $t_0 = -T$:

$$|x_0, -T\rangle = e^{-iHT} |x_0\rangle = \sum_n e^{-iHT} |n\rangle \langle n|x_0\rangle = |\Omega\rangle \langle \Omega|x_0\rangle + \sum_{n=1} e^{-iE_n T} |n\rangle \langle n|x_0\rangle.$$

In the limit $T \rightarrow \infty(1-i\epsilon)$ only the ground-state contribution will survive, and therefore we obtain for a generic operator O :

$$\lim_{T \rightarrow \infty(1-i\epsilon)} \langle x_N, T | O | x_0, -T \rangle = \langle x_N | \Omega \rangle \langle \Omega | x_0 \rangle \langle \Omega | O | \Omega \rangle. \quad (12.15)$$

The same expression with $O = 1$ involves $\langle \Omega | \Omega \rangle = 1$, and by dividing both we get rid of the prefactor on the r.h.s.:

$$\langle \Omega | T X(t_1) \dots X(t_n) | \Omega \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\int \mathcal{D}x e^{iS} x(t_1) \dots x(t_n)}{\int \mathcal{D}x e^{iS}}, \quad (12.16)$$

where the temporal integral in the action runs from $-T$ to T .

Path integrals for scalar fields.

Generating functionals.

Perturbation theory. We split the action into a free and an interacting part:

$$S[\Phi] = S_0[\Phi] + S_{\text{int}}[\Phi] \quad \Rightarrow \quad Z[J] = \int \mathcal{D}\Phi e^{iS_{\text{int}}[\Phi]} e^{iS_0[\Phi] + iJ \cdot \Phi}. \quad (12.17)$$

The trick is now to use the identities

$$\frac{\delta}{i\delta J(x)} e^{iJ \cdot \Phi} = \Phi(x) e^{iJ \cdot \Phi} \quad \text{and} \quad F \left[\frac{\delta}{i\delta J(x)} \right] e^{iJ \cdot \Phi} = F[\Phi(x)] e^{iJ \cdot \Phi}, \quad (12.18)$$

where F is some polynomial, to express $S_{\text{int}}[\Phi]$ through $S_{\text{int}}[\delta/i\delta J]$ and, since it no longer depends on the field, pull it out of the path integral:

$$Z[J] = e^{iS_{\text{int}} \left[\frac{\delta}{i\delta J(x)} \right]} \int \mathcal{D}\Phi e^{iS_0[\Phi] + iJ \cdot \Phi} = e^{iS_{\text{int}} \left[\frac{\delta}{i\delta J(x)} \right]} Z_0[J]. \quad (12.19)$$

The same trick would also work in reverse, i.e., we could have pulled out S_0 instead of S_{int} . The point is that we can work on the free part $Z_0[J]$ of the generating functional. We can write the free action as

$$S_0[\Phi] = \frac{1}{2} \Phi \cdot K \Phi, \quad K(x, y) = -\delta^4(x - y) (\square + m_0^2 - i\epsilon). \quad (12.20)$$

The kernel is the inverse of the scalar propagator, which is the Green function of the Klein-Gordon equation (cf. Eq. (2.77)):

$$\begin{aligned} \int d^4y K(x, y) K^{-1}(y, z) &= - \int d^4y \delta^4(x - y) (\square_x + m_0^2 - i\epsilon) K^{-1}(y, z) \\ &= -(\square_x + m_0^2 - i\epsilon) K^{-1}(x, z) = \delta^4(x - z). \end{aligned} \quad (12.21)$$

Due to the factor $-i\epsilon$ it is the **Feynman propagator**: $K(x, y) = iD_F^{-1}(x - y)$. This is what we referred to earlier when we claimed that the free action is just the inverse propagator.

We can further ‘complete the square’ by rewriting the exponent in Eq. (??) as

$$\frac{i}{2} \Phi \cdot K \Phi + i\Phi \cdot J = \frac{i}{2} (\Phi + K^{-1}J) \cdot K (\Phi + K^{-1}J) - \frac{i}{2} J \cdot K^{-1}J, \quad (12.22)$$

because partial integration allows us to set $\Phi \cdot K \chi = (K\Phi) \cdot \chi$. If we perform a shift of the field $\Phi \rightarrow \Phi + K^{-1}J$ inside the path integral, then the measure remains the same, $\mathcal{D}\Phi = \mathcal{D}\Phi'$, but the path integral becomes

$$Z_0[J] = \int \mathcal{D}\Phi' e^{\frac{i}{2}\Phi' \cdot K \Phi'} e^{-\frac{i}{2}J \cdot K^{-1}J} = Z_0[0] e^{\frac{1}{2}(iJ) \cdot D_F(iJ)}, \quad (12.23)$$

and in total:

$$\frac{Z[J]}{Z_0[0]} = e^{iS_{\text{int}}\left[\frac{\delta}{i\delta J}\right]} e^{\frac{1}{2}(iJ) \cdot D_F(iJ)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(iS_{\text{int}} \left[\frac{\delta}{i\delta J} \right] \right)^n e^{\frac{1}{2}(iJ) \cdot D_F(iJ)}. \quad (12.24)$$

This formula is all we need for doing perturbation theory: a generic n -point function is the functional derivative

$$G(x_1, \dots, x_n) = \frac{\delta}{i\delta J(x_1)} \cdots \frac{\delta}{i\delta J(x_n)} \frac{Z[J]}{Z_0[0]} \Big|_{J=0}, \quad (12.25)$$

but since we have completely absorbed the path integral into $Z_0[0]$ we are left with repeated functional derivatives with respect to the sources $J(x_i)$. Moreover, we can simply forget about $Z_0[0]$ because it will always drop out in the ratio $Z[J]/Z_0[0]$.

Let’s start with the zeroth order in perturbation theory, which reproduces the free theory. In that case we have just

$$\frac{Z[J]}{Z_0[0]} = e^{\frac{1}{2}(iJ) \cdot D_F(iJ)} =: \omega. \quad (12.26)$$

Similarly to our analysis in the interaction picture it is helpful to employ a diagrammatic notation. If we denote $iJ(x)$ by a blob and the Feynman propagator $D_F(x - y)$ by a line, we can write

$$\frac{1}{2} (iJ) \cdot D_F(iJ) = \frac{1}{2} \int d^4x \int d^4y iJ(x) D_F(x - y) J(y) = \frac{1}{2} \bullet \text{---} \bullet, \quad (12.27)$$

and a functional derivative amounts to

$$\frac{\delta}{i\delta J(x_1)} \frac{1}{2} (iJ) \cdot D_F(iJ) = \int d^4x D_F(x_1 - x) iJ(x) = 1 \text{---} \bullet. \quad (12.28)$$

If we further abbreviate $\delta/i\delta J(x_i) \equiv \delta_i$, then we can write the non-interacting one-

two-, three- and four-point functions (before setting all sources to zero) as

$$\begin{aligned}
 \delta_1 \omega &= \omega \left(1 \text{---} \bullet \right), \\
 \delta_1 \delta_2 \omega &= \omega \left(1 \text{---} 2 + \begin{array}{c} 1 \text{---} \bullet \\ 2 \text{---} \bullet \end{array} \right), \\
 \delta_1 \delta_2 \delta_3 \omega &= \omega \left(\begin{array}{c} 1 \text{---} 3 \\ 2 \text{---} \bullet \end{array} + \begin{array}{c} 2 \text{---} 3 \\ 1 \text{---} \bullet \end{array} + \begin{array}{c} 1 \text{---} 2 \\ 3 \text{---} \bullet \end{array} + \begin{array}{c} 1 \text{---} \bullet \\ 2 \text{---} \bullet \\ 3 \text{---} \bullet \end{array} \right), \\
 \delta_1 \delta_2 \delta_3 \delta_4 \omega &= \omega \left(\begin{array}{c} 1 \text{---} 3 \\ 2 \text{---} 4 \end{array} + \begin{array}{c} 2 \text{---} 3 \\ 1 \text{---} 4 \end{array} + \begin{array}{c} 1 \text{---} 2 \\ 3 \text{---} 4 \end{array} + \begin{array}{c} 1 \text{---} 4 \\ 2 \text{---} \bullet \\ 3 \text{---} \bullet \end{array} + \begin{array}{c} 1 \text{---} \bullet \\ 2 \text{---} 4 \\ 3 \text{---} \bullet \end{array} \right. \\
 &\quad \left. + \begin{array}{c} 1 \text{---} \bullet \\ 2 \text{---} \bullet \\ 3 \text{---} 4 \end{array} + \begin{array}{c} 1 \text{---} 3 \\ 2 \text{---} \bullet \\ 4 \text{---} \bullet \end{array} + \begin{array}{c} 2 \text{---} 3 \\ 1 \text{---} \bullet \\ 4 \text{---} \bullet \end{array} + \begin{array}{c} 1 \text{---} 2 \\ 3 \text{---} \bullet \\ 4 \text{---} \bullet \end{array} + \begin{array}{c} 1 \text{---} \bullet \\ 2 \text{---} \bullet \\ 3 \text{---} \bullet \\ 4 \text{---} \bullet \end{array} \right).
 \end{aligned}$$

Setting the sources to zero has the same effect as retaining only complete Wick contractions, and so we are left with

$$\begin{aligned}
 G(x_1, x_2) &= D_F(x_1 - x_2) = 1 \text{---} 2, \\
 G(x_1 \dots x_4) &= \begin{array}{c} 1 \text{---} 2 \\ 3 \text{---} 4 \end{array} + \begin{array}{c} 1 \\ | \\ 3 \end{array} \begin{array}{c} 2 \\ | \\ 4 \end{array} + \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ 3 \quad 4 \end{array}, \tag{12.29}
 \end{aligned}$$

just like earlier.

To first order perturbation theory we additionally have to apply the term S_{int} to ω :

$$iS_{\text{int}} \left[\frac{\delta}{i\delta J} \right] \omega = -\frac{i\lambda}{4!} \int d^4z \frac{\delta^4}{i\delta J(z)^4} \omega = -\frac{i\lambda}{4!} \int d^4z \delta_z^4 \omega. \tag{12.30}$$

If we set $z = x_1 = x_2 = x_3 = x_4$ and identify the vertex with $-i\lambda$, we can simply read off the result from the last line above:

$$-\frac{i\lambda}{4!} \int d^4z \delta_z^4 \omega = \frac{\omega}{4!} \left[3 \text{---} \bullet \text{---} \bullet + 6 \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} \right]. \tag{12.31}$$

Before calculating n -point functions, let's put this back into Eq. (??) for the generating functional:

$$\frac{Z[J]}{Z_0[0]} = \omega \left[1 + \frac{1}{8} \text{---} \bullet \text{---} \bullet + \frac{1}{4} \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} + \frac{1}{24} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} \right], \quad \frac{Z[0]}{Z_0[0]} = 1 + \frac{1}{8} \text{---} \bullet \text{---} \bullet.$$

As we announced earlier around Eq. (??), the generating functional is the sum of all **vacuum bubbles** together with their symmetry factors. On the other hand, these vacuum bubbles *cancel* when we take the ratio

$$\frac{Z[J]}{Z[0]} = \omega \left[1 + \frac{1}{4} \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} + \frac{1}{24} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} \right] \tag{12.32}$$

and thereby do not contribute to the n -point functions.

To calculate Green functions we have to apply further derivatives. In the end all sources will be set to zero, so if we are only interested in the propagator we can ignore all diagrams with more than two source terms:

$$\begin{aligned} \delta_1 \frac{Z[J]}{Z[0]} &= \omega \left[\frac{1}{2} \text{diagram}_1 + 1 \text{diagram}_2 + \text{diagrams with } \geq 2 \text{ sources} \right], \\ \delta_1 \delta_2 \frac{Z[J]}{Z[0]} &= \omega \left[\frac{1}{2} \text{diagram}_3 + 1 \text{diagram}_4 + \text{diagrams with } \geq 1 \text{ source} \right]. \end{aligned} \quad (12.33)$$

In total we arrive at the same result for the propagator up to $\mathcal{O}(\lambda)$ as earlier:

$$G(x_1, x_2) = \text{diagram}_1 + \text{diagram}_2. \quad (12.34)$$

If we repeat the procedure up to $\mathcal{O}(\lambda^2)$ both for the propagator and the four-point function, we will generate all partially connected diagrams from Eqs. (??) and (??).

Connected and 1PI diagrams. The language with source terms also allows us to construct generating functionals for the connected and 1PI diagrams. The generating functional for **connected diagrams** is the logarithm of $Z[J]/Z[0]$:

$$G(x_1, \dots, x_n)_{\text{conn}} = \frac{\delta}{i\delta J(x_1)} \cdots \frac{\delta}{i\delta J(x_n)} \Big|_{J=0} iW[J], \quad iW[J] = \ln \frac{Z[J]}{Z[0]}, \quad (12.35)$$

which can be seen as follows. The full n -point functions (including partially disconnected terms) are obtained by taking the derivatives of $Z[J]/Z[0] = e^{iW[J]}$. We abbreviate once more

$$\delta_i iW[J] \Big|_{J=0} = w_i = 0, \quad \delta_i \delta_j iW[J] \Big|_{J=0} = w_{ij}, \quad \text{etc.}, \quad (12.36)$$

where w_i is the connected one-point function (which is zero), w_{ij} the connected two-point function etc. Then we find

$$\begin{aligned} \delta_1 \delta_2 \frac{Z[J]}{Z[0]} \Big|_{J=0} &= w_{12}, \\ \delta_1 \delta_2 \delta_3 \delta_4 \frac{Z[J]}{Z[0]} \Big|_{J=0} &= w_{1234} + w_{12} w_{34} + w_{13} w_{24} + w_{14} w_{23}, \end{aligned} \quad (12.37)$$

and so on, which are indeed the correct relations between the full and connected n -point functions. We can easily verify this for the example in Eq. (12.32):

$$iW[J] = \ln \frac{Z[J]}{Z[0]} = \frac{1}{2} \text{diagram}_1 + \frac{1}{4} \text{diagram}_2 + \frac{1}{24} \text{diagram}_3 + \mathcal{O}(\lambda^2). \quad (12.38)$$

Taking derivatives is now much easier because we don't need to worry about the contribution from ω . In fact, it is so simple that we can already read off the result for the two-point function (which is the propagator itself) and the four-point function (which contains the connected diagram only).

To construct the generating functional for 1PI Green functions, we first define the ‘**averaged field**’ $\varphi(x)$ as the vacuum expectation value of $\Phi(x)$ for a nonvanishing source:

$$\varphi(x) := \frac{\delta W[J]}{\delta J(x)} = \frac{\delta}{i\delta J(x)} \ln \frac{Z[J]}{Z[0]} = \frac{1}{Z[J]} \frac{\delta}{i\delta J(x)} Z[J] = \langle \Omega | \Phi(x) | \Omega \rangle_J. \quad (12.39)$$

That is, $\varphi(x)$ is the 1-point function in the presence of the source J ; it vanishes if we set $J = 0$ because the vacuum expectation value of the field vanishes.¹⁰ Now assume that this relation is invertible, so that for a given $\varphi(x)$ there exists a unique $J(x)$ that depends on $\varphi(x)$.¹¹ We can then define the **effective action**, which is a functional of $\varphi(x)$ only, through the Legendre transformation of $W[J]$:

$$\Gamma[\varphi] := W[J] - \int d^4y J(y) \varphi(y). \quad (12.40)$$

This leads to an analogous relation for $J(x)$:

$$\frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} = \int d^4y \underbrace{\frac{\delta W[J]}{\delta J(y)}}_{\varphi(y)} \frac{\delta J(y)}{\delta \varphi(x)} - J(x) - \int d^4y \frac{\delta J(y)}{\delta \varphi(x)} \varphi(y) = -J(x). \quad (12.41)$$

Therefore, the sources $J(x)$ and $\varphi(x)$ are conjugated,

$$\frac{\delta W[J]}{\delta J(x)} = \varphi(x), \quad \frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} = -J(x), \quad (12.42)$$

similarly to thermodynamic systems: if we were to interpret $W[J]$ and $\Gamma[\varphi]$ as thermodynamic potentials, then $J(x)$ would be the ‘intensive’ and $\varphi(x)$ the ‘extensive’ variable, and differentiation with respect to one variable gives the conjugated one.¹² To better understand the meaning of ‘effective action’, recall that

$$\frac{Z[J]}{Z[0]} = \frac{\int \mathcal{D}\Phi e^{iS[\Phi] + iJ \cdot \Phi}}{\int \mathcal{D}\Phi e^{iS[\Phi]}} = e^{iW[J]} = e^{i\Gamma[\varphi] + J \cdot \varphi}. \quad (12.43)$$

That is, the averaged field is the VEV of the classical field, and the effective action is the quantum averaged action, integrated over quantum fluctuations, with the path integral exponential as weight factor. In the same way as the classical action $S[\varphi]$ contains the full content of the classical field theory, either of the functionals $Z[J]$,

¹⁰By Lorentz invariance $\langle \Omega | \Phi(x) | \Omega \rangle = \langle \Omega | \Phi(0) | \Omega \rangle = c$, and if $c \neq 0$ we can still redefine the field by a shift $\Phi(x) \rightarrow \Phi(x) - c$ so that $\varphi(x)_{J=0} = 0$.

¹¹This is indeed true in the Euclidean formulation where $W[J]$ is a convex function.

¹²The analogy between Euclidean quantum field theory and statistical physics can be taken further: finite \hbar lead to quantum fluctuations and finite $k_B T$ to thermal fluctuations, which are encoded in the effective action.

$W[J]$ or $\Gamma[\varphi]$ determines the quantum field theory completely since all Green functions can be derived from them.

Our goal in the following is to show that the effective action generates all **1-particle irreducible Green functions**:

$$\Gamma_n(x_1, \dots, x_n) = \frac{\delta}{\delta\varphi(x_1)} \cdots \frac{\delta}{\delta\varphi(x_n)} i\Gamma[\varphi] \Big|_{\varphi=0}. \quad (12.44)$$

Path integral for fermions. The path integral quantization has a convenient side effect: all fields are numbers instead of operators, so in principle we could completely forget about the operator structure, the equal-time commutation relations, etc. The Green functions are time-ordered VEVs of operators, but at the same time the quantum averages of the classical fields. There is one problem, though: what is the classical counterpart of *anticommuting* fermion field operators?

The solution is that we must treat them as anticommuting **Grassmann variables**. Consider an n -dimensional vector space with basis elements $\theta_1, \dots, \theta_n$, i.e., their addition and multiplication with scalars is defined. Since we also want to multiply the θ_i among themselves, we impose the anticommutator relation

$$\{\theta_i, \theta_j\} = 0 \quad (12.45)$$

which defines the **Grassmann algebra**. The fact that by this definition $\theta_i \theta_j = -\theta_j \theta_i$ and $\theta_i^2 = 0$ has some interesting consequences. First, the most general element of the algebra has the form

$$f(\theta_1 \dots \theta_n) = c + \sum_i c_i \theta_i + \frac{1}{2!} \sum_{ij} c_{ij} \theta_i \theta_j + \cdots + \frac{1}{n!} \sum_{i_1 \dots i_n} c_{i_1 \dots i_n} \theta_{i_1} \dots \theta_{i_n}, \quad (12.46)$$

with totally antisymmetric coefficients $c_{i_1 \dots i_n}$. Alternatively, we could omit the factorial and sum over $i_1 < \dots < i_n$ only; the last term is simply $c_{1 \dots n} \theta_1 \dots \theta_n$. In any case, the expansion *stops* because the dependence on each θ_i can be at most linear. For example in one or two dimensions:

$$f(\theta) = c + c_1 \theta, \quad f(\theta_1, \theta_2) = c + c_1 \theta_1 + c_2 \theta_2 + c_{12} \theta_1 \theta_2, \quad (12.47)$$

and a Taylor expansion has the form $e^{a\theta} = 1 + a\theta$. Since each θ_i appears at most linearly, applying a derivative has the effect of replacing $\theta_i \rightarrow 1$; however, one first has to permute θ_i towards the derivative operator:

$$\frac{\partial f(\theta)}{\partial \theta} = c_1, \quad \frac{\partial f(\theta_1, \theta_2)}{\partial \theta_1} = c_1 + c_{12} \theta_2, \quad \frac{\partial f(\theta_1, \theta_2)}{\partial \theta_2} = c_2 - c_{12} \theta_1 \quad (12.48)$$

and

$$\frac{\partial^2 f(\theta_1, \theta_2)}{\partial \theta_1 \partial \theta_2} = -\frac{\partial^2 f(\theta_1, \theta_2)}{\partial \theta_2 \partial \theta_1} = -c_{12}. \quad (12.49)$$

Next, we *define* the integration as

$$\int d\theta 1 = 0, \quad \int d\theta \theta = 1. \quad (12.50)$$

The first relation follows from the requirement that translation invariance for a convergent bosonic integral should also hold for an integral over Grassmann variables:

$$\int d\theta f(\theta) = \int d\theta f(\theta + \eta) \quad \Rightarrow \quad \int d\theta \eta = 0, \quad (12.51)$$

and the second is a normalization convention. In addition we require that the integration variable must be permuted to the integral measure, like for the derivative. Consequently, the integration has the same effect as a derivative:

$$\int d\theta f(\theta) = \int d\theta (c + c_1 \theta) = c_1 = \frac{df(\theta)}{\partial\theta}. \quad (12.52)$$

If we define the Grassmann measure as $d^n\theta = d\theta_n \dots d\theta_1$, then

$$\int d^n\theta \theta_1 \dots \theta_n = 1, \quad \int d^n\theta \theta_{i_1} \dots \theta_{i_n} = \varepsilon_{i_1 \dots i_n}, \quad (12.53)$$

where $\varepsilon_{i_1 \dots i_n}$ is the totally antisymmetric tensor normalized to $\varepsilon_{1 \dots n} = 1$. Therefore, in a generic integral only the last term in Eq. (12.46) survives because the number of Grassmann variables it contains must be saturated by the integration measure:

$$\int d^n\theta f(\theta_1 \dots \theta_n) = \int d^n\theta c_{1 \dots n} \theta_1 \dots \theta_n = c_{1 \dots n}. \quad (12.54)$$

For the same reason an integration by parts does not pick up any boundary terms because the integral of a derivative vanishes:

$$\int d^n\theta \frac{\partial}{\partial\theta_i} f(\theta_1 \dots \theta_n) = 0. \quad (12.55)$$

From these relation one can show that the Jacobian that corresponds to the transformation $\theta'_i = A_{ij} \theta_j$ is the *inverse* determinant. Abbreviating $\{\theta_1 \dots \theta_n\} \equiv \theta$,

$$\begin{aligned} \int d^n\theta f(\theta') &= \int d^n\theta c_{1 \dots n} A_{1i_1} \dots A_{ni_n} \theta_{i_1} \dots \theta_{i_n} \\ &= \underbrace{\varepsilon_{i_1 \dots i_n} A_{1i_1} \dots A_{ni_n}}_{\det A} c_{1 \dots n} = \det A \int d^n\theta f(\theta) = \det A \int d^n\theta' f(\theta'), \end{aligned} \quad (12.56)$$

and therefore $d^n\theta' = (\det A)^{-1} d^n\theta$. Compare this to the bosonic case, where for $x'_i = A_{ij} x_j$ we have $d^n x' = |\det A| d^n x$.

One can generalize these formulas to complex Grassmann variables, where the complex conjugation is defined by $(\theta_i \theta_j)^* = \theta_j^* \theta_i^*$ and the integral measure by

$$d^n\theta d^n\theta^* = d\theta_1 d\theta_1^* \dots d\theta_n d\theta_n^* = d\theta_n d\theta_n^* \dots d\theta_1 d\theta_1^*. \quad (12.57)$$

The generalization of Eq. (??) is

$$\int d^n\theta d^n\theta^* (\theta_{i_1}^* \theta_{j_1} \dots \theta_{i_n}^* \theta_{j_n}) = \varepsilon_{i_1 \dots i_n} \varepsilon_{j_1 \dots j_n}. \quad (12.58)$$

For a generic integral

$$\int d^n \theta d^n \theta^* f(\theta, \theta^*) \quad (12.59)$$

only the term which is proportional to $\theta_1 \dots \theta_n \theta_1^* \dots \theta_n^*$ contributes. Under a unitary transformation the Jacobian is $(\det U)(\det U)^* = 1$ and therefore unitary transformations do not change the integral.

One often deals with Gaussian integrals, where the integrand is $e^{\theta_i^* B_{ij} \theta_j}$. In the series expansion only the term of order n survives because it has the same number of Grassmann variables as the integral measure:

$$\begin{aligned} \int d^n \theta d^n \theta^* e^{\theta_i^* B_{ij} \theta_j} &= \int d^n \theta d^n \theta^* \frac{1}{n!} (\theta_i^* B_{ij} \theta_j)^n \\ &= \frac{1}{n!} B_{i_1 j_1} \dots B_{i_n j_n} \int d^n \theta d^n \theta^* (\theta_{i_1}^* \theta_{j_1} \dots \theta_{i_n}^* \theta_{j_n}) \\ &= \frac{1}{n!} \varepsilon_{i_1 \dots i_n} \varepsilon_{j_1 \dots j_n} B_{i_1 j_1} \dots B_{i_n j_n} = \det B. \end{aligned} \quad (12.60)$$

If the exponent comes with a minus sign, one picks up a factor $(-1)^n$ in the Taylor expansion which can be compensated by interchanging $d^n \theta d^n \theta^* \rightarrow d^n \theta^* d^n \theta$. Compare these results again with the bosonic case:

$$\begin{aligned} \int d^n \theta^* d^n \theta e^{-\theta_i^* B_{ij} \theta_j} &= \det B \quad (\text{fermionic}), \\ \int d^n x^* d^n x e^{-x_i^* B_{ij} x_j} &= \frac{(2\pi)^n}{\det B} \quad (\text{bosonic}). \end{aligned} \quad (12.61)$$

If the variables are real, we have to take the square root instead:

$$\begin{aligned} \int d^n \theta e^{\frac{1}{2} \theta_i B_{ij} \theta_j} &= \sqrt{\det B} \quad (\text{fermionic}), \\ \int d^n x e^{-\frac{1}{2} x_i B_{ij} x_j} &= \sqrt{\frac{(2\pi)^n}{\det B}} \quad (\text{bosonic}). \end{aligned} \quad (12.62)$$

With that we have everything in place to write down the **path integral for fermions**. The fermion fields $\psi(x)$ and $\bar{\psi}(x)$ are now Grassmann-valued fields; for example, one could write them in a given basis as

$$\psi(x) = \sum_i \psi_i \phi_i(x). \quad (12.63)$$

The $\phi_i(x)$ are ordinary commuting fields (for example a basis of Dirac spinors), whereas the coefficients ψ_i are Grassmann numbers. To arrive at the generating functional analogous to Eq. (??), one introduces Grassmann-valued source terms $\eta(x)$ and $\bar{\eta}(x)$:

$$Z_0[\bar{\eta}, \eta] = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{i \int d^4 x [\bar{\psi} (i\cancel{\partial} - m_0) \psi + \bar{\eta} \psi + \bar{\psi} \eta]}. \quad (12.64)$$

Following the same steps as in Eq. (??) leads to

$$Z_0[\bar{\eta}, \eta] = Z_0[0, 0] e^{(i\bar{\eta}) \cdot S_F(i\eta)}, \quad (12.65)$$

from where the Green functions of the free Dirac theory can be calculated. Since $\bar{\eta}(x)$ and $\eta(x)$ are Grassmann fields, however, we must be careful when taking derivatives because the respective field must be first permuted to the left. For example, the two-point function is given by

$$\langle 0 | \mathbb{T} \bar{\psi}_\alpha(x_1) \psi_\beta(x_2) | 0 \rangle = \left(\frac{\delta}{i \delta \bar{\eta}_\alpha(x_1)} \right) \left(\frac{\delta}{-i \delta \eta_\beta(x_2)} \right) \frac{Z_0[\bar{\eta}, \eta]}{Z_0[0, 0]} \Big|_{\bar{\eta}=\eta=0}. \quad (12.66)$$

DSEs and WTIs.

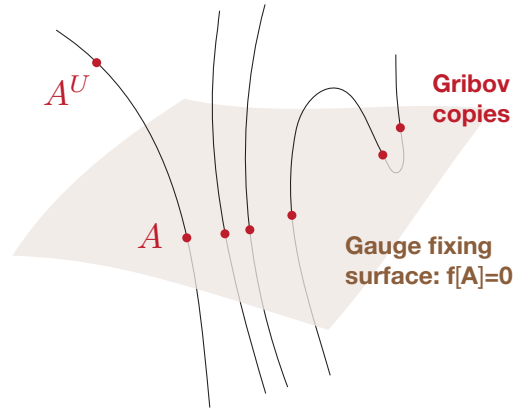


FIGURE 13.1: Gauge orbits and gauge-fixing surface.

13 Non-Abelian gauge theories

[Unfinished]

Local gauge invariance revisited.

Non-Abelian gauge theories.

The Standard Model.

Faddeev-Popov quantization. The standard method is the Faddeev-Popov gauge fixing procedure. Let's denote a gauge transformation of the gluon field by $A \rightarrow A^U$, where U is some gauge transformation with gauge parameter ε . Impose a gauge-fixing function $f[A]$ which we want to set to zero at the end: $f[A] = 0$, to single out a hypersurface of fixed gauge (cf. Fig. 13.1).

The basic idea is to restrict the path integral to a hypersurface that satisfies the gauge-fixing condition by inserting some unity operator:

$$\int \mathcal{D}A e^{iS} = \int \mathcal{D}A \underbrace{\delta(f[A]) (\dots)}_{=1} e^{iS}. \quad (13.1)$$

To this end, consider a one-dimensional function $f(\varepsilon)$. It satisfies the identity

$$\int_{-\infty}^{\infty} d\varepsilon \left| \frac{df(\varepsilon)}{d\varepsilon} \right|_{f(\varepsilon)=0} \delta(f(\varepsilon)) = \int_{-\infty}^{\infty} d\varepsilon |f'(\varepsilon_0)| \frac{\delta(\varepsilon - \varepsilon_0)}{|f'(\varepsilon_0)|} = 1, \quad (13.2)$$

By means of the δ -function, ε_0 is the value where $f(\varepsilon) = 0$; if $f(\varepsilon)$ has several zeros we would have to sum over them. The infinite-dimensional continuum version of this relation is the 'functional unity'

$$\int \mathcal{D}U \det M[A] \delta(f[A^U]) = 1, \quad (13.3)$$

where the **Faddeev-Popov operator** $M[A]$ is the derivative of the gauge-fixing condition with respect to the gauge transformation parameter,

$$M[A] := \left. \frac{\delta f[A^U]}{\delta \varepsilon} \right|_{f[A^U]=0}. \quad (13.4)$$

The δ -function is an infinite product of δ -functions at each space-time point x , and $\mathcal{D}U$ is called the group measure. $M[A]$ does not depend on the gauge transformation U : for example, a linear covariant gauge is defined by $f[A] = \partial_\mu A^\mu$, and from Eq. (??) we have $\delta A^\mu = \frac{1}{g} D^\mu \varepsilon$, so that the Faddeev-Popov operator in this case is given by

$$M[A]_{ab}(x, y) = \frac{1}{g} \partial_\mu D_{ab}^\mu \delta^4(x - y). \quad (13.5)$$

In QED, this expression is also independent of A so we can pull it out of the path integral.¹³

We can insert Eq. (13.3) in the path integral:

$$Z = \int \mathcal{D}U \int \mathcal{D}A \det M[A] \delta(f[A^U]) e^{iS[A]}, \quad (13.6)$$

and, since Z is gauge-invariant, perform a gauge transformation $A^U \rightarrow A$. The gauge field measure $\mathcal{D}A$, the group measure $\mathcal{D}U$, the Faddeev-Popov determinant and the classical action $S[A]$ are all invariant under this operation, so that it merely amounts to replacing $\delta(f[A^U]) \rightarrow \delta(f[A])$. The integrand then depends no longer on U and the group integration $\mathcal{D}U$ factorizes; it produces an infinite constant which drops out whenever we normalize Z , for example when calculating Green functions. The remaining δ -function restricts the integration over all fields to the hypersurface $f[A] = 0$. Each gauge orbit contributes only one field configuration and we have an integration over physically distinct fields. In total we have indeed arrived at the form (??), where the bracket with dots is the Faddeev-Popov determinant.

Recall that a determinant can be written as a Gaussian integral over Grassmann variables. A *functional* determinant is therefore the Gaussian *path* integral over Grassmann *fields*:

$$\det M[A] = \int \mathcal{D}c \mathcal{D}\bar{c} e^{-\iint \bar{c}_a(x) M[A]_{ab}(x, y) c_b(y)}, \quad (13.7)$$

where the Faddeev-Popov ghosts $c^a(x)$, $\bar{c}^a(x)$ are scalar but Grassmann-valued fields. They carry the wrong Bose-Fermi statistics, but this is of no concern since they are unphysical anyway. In conclusion, we have found that introducing a gauge-fixing term also introduces ghost fields.¹⁴ This leads to new Green functions; from Eq. (13.7) we see that the Faddeev-Popov operator defines the inverse tree-level ghost propagator.

¹³Here we have assumed that the gauge-fixing condition is unique, i.e., that the equation $f(\varepsilon) = 0$ admits only one solution ε_0 . This is usually not the case due to Gribov copies: the gauge-fixing condition can intersect the gauge orbits more than once and is therefore not complete. In this case both $f[A] = 0$ and $f[A^U] = 0$ are realized on the same gauge orbit, which means that the FP operator has zero eigenvalues. The problem does not appear in QED where the residual gauge freedom can be removed by imposing appropriate boundary conditions on the fields.

¹⁴In QED (at least with linear gauges), the ghosts are not dynamical because the Faddeev-Popov determinant is A -independent and can be pulled out of the path integral.

The remaining goal is to shuffle the Faddeev-Popov determinant and the δ -function in (13.6) into the action, at the price of introducing new, unphysical fields which are merely a consequence of fixing the gauge. We can take care of the δ -function by changing the gauge fixing condition to $f[A] + \frac{\xi}{2} B = 0$, where $B(x)$ lives in the Lie algebra but does not depend on A . This does not affect the Faddeev-Popov determinant, but the functional integral Z_B depends now implicitly on B . Since any B leads to the same gauge-invariant physics, we can work with Z_B , $Z_{B'}$ or $\int \mathcal{D}B F(B) Z_B$; these are all equivalent. If we integrate over the functions $B(x)$ with some Gaussian weight, we can remove the δ -function in favor of a new term in the action:

$$\begin{aligned} Z &= \int \mathcal{D}B e^{-\frac{i\xi}{8} \int d^4x B^2(x)} \mathcal{D}A \det M[A] \delta(f[A] + \frac{\xi}{2} B) e^{iS[A]} \\ &= \int \mathcal{D}A \det M[A] e^{i\left(S[A] - \int d^4x \frac{f[A]^2}{2\xi}\right)}. \end{aligned} \quad (13.8)$$

For example, with a linear covariant gauge this provides a welcome modification to the propagator term in Eq. (??), as it is no longer transverse in momentum space but has instead the form

$$-\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} - \frac{(\partial_\mu A_a^\mu)^2}{2\xi} \cong \frac{1}{2} A_\mu^a \left(\square g^{\mu\nu} - \partial^\mu \partial^\nu + \frac{1}{\xi} \partial^\mu \partial^\nu \right) A_\nu^a + \dots \quad (13.9)$$

and can be inverted. ξ is the gauge parameter: $\xi = 0$ defines the Landau gauge, $\xi = 1$ the Feynman gauge, and there are many other possible choices which differ not only by the gauge parameter but also by the gauge fixing condition (Coulomb gauge, axial gauge, light-cone gauge, maximal Abelian gauge etc.).

Reinserting the quarks and including all source terms, the final partition function for QCD assumes the form

$$Z[J, \eta, \bar{\eta}, \sigma, \bar{\sigma}] = \int \mathcal{D}[A, \psi, \bar{\psi}, c, \bar{c}] e^{i(S[A, \psi, \bar{\psi}] + S_{\text{GF}}[A, c, \bar{c}] + S_{\text{C}})}, \quad (13.10)$$

where the gauge-fixing part of the action is

$$S_{\text{GF}} = - \int_x \frac{f[A]^2}{2\xi} + \iint_{x y} i \bar{c}_a M[A]_{ab} c_b = \int_x \left(- \frac{(\partial_\mu A_a^\mu)^2}{2\xi} + \frac{i}{g} \bar{c}_a \partial_\mu D_{ab}^\mu c_b \right). \quad (13.11)$$

The second equality holds for a linear covariant gauge; the factor i/g can be absorbed in the ghost fields. The source term reads

$$S_{\text{C}} = - \int_x (J_\mu A^\mu + \bar{\psi} \eta + \bar{\eta} \psi + \bar{\sigma} c + \bar{c} \sigma), \quad (13.12)$$

where $\eta, \bar{\eta}$ are the quark sources and $\sigma, \bar{\sigma}$ the ghost sources.

An equivalent way to arrive at Eq. (13.11) is to impose **BRST invariance** of the action (Becchi, Rouet, Stora, Tyutin). Consider an infinitesimal gauge transformation (??) where the gauge parameter is a ghost field $c(x) = c_a(x) \mathbf{t}_a$, i.e., a scalar anticommuting Grassmann field:

$$\delta\psi = ic\psi, \quad \delta\bar{\psi} = -i\bar{\psi}c, \quad \delta A_\mu = \frac{1}{g} D_\mu c, \quad \delta F_{\mu\nu} = i[c, F_{\mu\nu}]. \quad (13.13)$$

If we further demand that this transformation be nilpotent ($\delta^2 = 0$), it is straightforward to prove that any of the relations above fixes the transformation behavior of the ghost itself: $\delta c = \frac{1}{2} [c, c]$ or, in components: $\delta c_a = -\frac{1}{2} f_{abc} c_b c_c$. (Nilpotency of this last relation can be shown using the Jacobi identity; note that δ also anticommutes with c). Applying δ increases the ghost number (the charge corresponding to a $U(1)$ symmetry of the ghost fields) by one unit; hence, when applied to the antighost, it must produce a scalar field with ghost number zero, the so-called Nakanishi-Lautrup field: $\delta \bar{c} = -iB$. Nilpotency of the antighost transformation then fixes $\delta B = 0$. The different treatment of c and \bar{c} implies that they are not conjugates of each other but truly independent fields.

Since the classical action $S[A, \psi, \bar{\psi}]$ is gauge invariant and BRST is a gauge transformation, it is also BRST invariant. The most general BRST-invariant action is then the sum of the classical action plus a term $S_{\text{GF}} = \delta \mathcal{O}$ which is a BRST variation itself, since in that case we have $\delta S_{\text{GF}} = \delta^2 \mathcal{O} = 0$. Adding this to the action means fixing a gauge; which gauge we get depends on \mathcal{O} . To recover (13.11), we contract the antighost with our earlier gauge-fixing condition $f[A] + \frac{\xi}{2} B$:

$$S_{\text{GF}} = \delta \int_x i \bar{c}_a \left(f_a[A] + \frac{\xi}{2} B_a \right) = \int_x B_a \left(f_a[A] + \frac{\xi}{2} B_a \right) + \int_x \int_y i \bar{c}_a M[A]_{ab} c_b. \quad (13.14)$$

Inserting the equations of motion for B_a , namely $f_a + \xi B_a = 0$, yields again Eq. (13.11); the same result follows from integrating over B_a in the path integral. Hence, imposing BRST invariance simultaneously generates gauge-fixing and ghost terms in the action.

Quantum Chromodynamics. We can carry over the same analysis from φ^4 theory to QCD. QCD is a renormalizable quantum field theory because its coupling g is dimensionless. We have now several distinct fields in the Lagrangian, defined by (??) plus the gauge-fixing part in (13.11), which we reinterpret as 'bare' quantities. Their relationship with the renormalized quantities introduces renormalization constants:

$$\psi_{\text{B}} = Z_\psi^{1/2} \psi, \quad A_{\text{B}} = Z_A^{1/2} A, \quad c_{\text{B}} = Z_c^{1/2} c, \quad m_{\text{B}} = Z_m m, \quad g_{\text{B}} = Z_g g. \quad (13.15)$$

From Eqs. (??), (??) and (13.11), the resulting Lagrangian would read explicitly (modulo partial integrations):¹⁵

$$\begin{aligned} \mathcal{L} = & Z_\psi \bar{\psi} (i \not{\partial} - Z_m \mathbf{M}) \psi + \frac{1}{2} A_\mu^a [Z_A (\square g^{\mu\nu} - \partial^\mu \partial^\nu) + \lambda \partial^\mu \partial^\nu] A_\nu^a + Z_c \bar{c}_a \square c_a \\ & + Z_\Gamma g \bar{\psi} \not{A} \psi + \tilde{Z}_\Gamma i g (\partial_\mu \bar{c}_a) [A^\mu, c]_a \\ & - Z_{3g} \frac{g}{2} f_{abc} (\partial^\mu A_a^\nu - \partial^\nu A_a^\mu) A_\mu^b A_\nu^c - Z_{4g} \frac{g^2}{4} f_{abe} f_{cde} A_a^\mu A_b^\nu A_\mu^c A_\nu^d. \end{aligned} \quad (13.16)$$

The first line contains the inverse tree-level quark, gluon and ghost propagators. The transversality condition for the gluon self-energy is the same as in QED, which entails that the longitudinal part of the gluon propagator stays unrenormalized. The second line contains the tree-level quark-gluon and ghost-gluon vertex and the third line the three- and four-gluon vertex. The vertex renormalization constants are related to those in Eq. (13.15) via

$$Z_\Gamma = Z_g Z_A^{1/2} Z_\psi, \quad \tilde{Z}_\Gamma = Z_g Z_A^{1/2} Z_c, \quad Z_{3g} = Z_g Z_A^{3/2}, \quad Z_{4g} = Z_g^2 Z_A^2. \quad (13.17)$$

In total, there are five independent renormalization constants ($Z_\psi, Z_A, Z_c, Z_m, Z_g$) and we must employ five renormalization conditions to fix them and remove all divergences from the theory.

¹⁵We have also rescaled the ghost fields to get rid of the coupling g in the denominator.

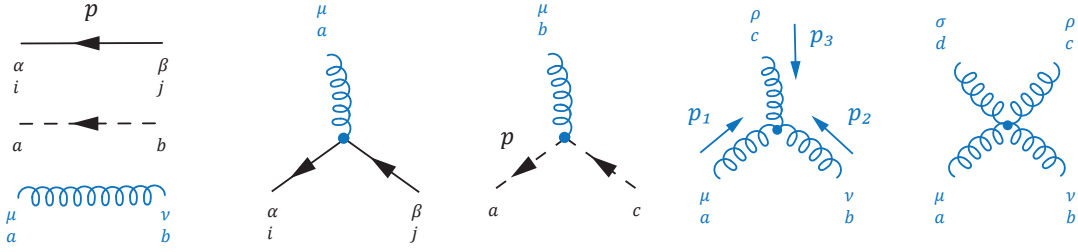


FIGURE 13.2: Tree-level propagators and vertices in the QCD action.

We have now everything in place to write down the final expressions for the renormalized tree-level propagators and vertices in QCD. They are necessary for perturbative calculations since the dressed n -point functions at large momenta revert to these forms, but they also enter as inputs for nonperturbative studies. The Feynman rules for the quark, gluon and ghost propagator are given by

$$\begin{aligned} iS_0^{-1}(p) &= Z_\psi (\not{p} - m_B), & i(D_0^{-1})^{\mu\nu}(q) &= -q^2 (Z_A T_q^{\mu\nu} + \lambda L_q^{\mu\nu}), \\ iG_0^{-1}(q) &= -Z_c q^2, \end{aligned} \quad (13.18)$$

where $m_0 = Z_m m$ and m is the renormalized quark mass. We abbreviated the longitudinal and transverse projectors that appear in the gluon propagator by $L_p^{\mu\nu} = p^\mu p^\nu / p^2$ and $T_p^{\mu\nu} = g^{\mu\nu} - L_p^{\mu\nu}$. Notice the ‘wrong’ sign for the ghost propagator; it tells us that ... The tree-level quark-gluon and ghost-gluon vertices read (see Fig. 13.2 for the kinematics)

$$\Gamma_{q,0}^\mu = ig Z_\Gamma \gamma^\mu \mathbf{t}_a, \quad \Gamma_{gh,0}^\mu = g \tilde{Z}_\Gamma f_{abc} p^\mu, \quad (13.19)$$

and the three- and four-gluon vertices are given by

$$\begin{aligned} \Gamma_{3g,0}^{\mu\nu\rho} &= g Z_{3g} f_{abc} \left[(p_1 - p_2)^\rho g^{\mu\nu} + (p_2 - p_3)^\mu g^{\nu\rho} + (p_3 - p_1)^\nu g^{\rho\mu} \right], \\ \Gamma_{4g,0}^{\mu\nu\rho\sigma} &= -ig^2 Z_{4g} \left[f_{abe} f_{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\nu\rho} g^{\mu\sigma}) + f_{ace} f_{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\nu\rho} g^{\mu\sigma}) \right. \\ &\quad \left. + f_{ade} f_{cbe} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\nu} g^{\rho\sigma}) \right]. \end{aligned} \quad (13.20)$$

QCD perturbation theory.

14 References

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