Optimal control and inverse problems

Christian Clason[1] **and Barbara Kaltenbacher**[2]

[1] Faculty of Mathematics, University of Duisburg-Essen, 45117 Essen, Germany

[2] Institute of Mathematics, Alpen-Adria Universität Klagenfurt, Universitätsstrasse 65–67, 9020 Klagenfurt, Austria

E-mail: barbara.kaltenbacher@aau.at

Optimal control of differential equations is concerned with finding "controls" (i.e., inputs such as right-hand sides, boundary conditions, coefficients, or domains) to differential equations such that their solution or "state" best approximates a given desired state by minimising a suitable functional, possibly under constraints on the control and/or the state. In general, such functionals can be split into a "tracking term" measuring the discrepancy of the optimal state to the desired state and "control costs" penalising the optimal control, linked through the (partial) differential equation as an equality constraint. (For this reason, such problems are also known as "(P)DE-constrained optimisation problems".) Typical questions, for a given functional and differential equation, are on the existence of minimisers, their stability with respect to perturbations of the data, the differentiability of the functional with respect to the control, and numerical methods for the (approximate) computation of optimal controls. For details, we refer to the seminal textbook [1] as well as to [2, 3] in particular regarding numerical methods and concrete applications.

Hence there are clear parallels with variational Tikhonov-, Morozov-, or Ivanov-regularisation of parameter identification problems for differential equations, both regarding well-posedness and, in particular, numerical methods. Both fields rely for this on an infinite-dimensional functional-analytic framework as guidance for accurate and efficient finite-dimensional numerical algorithms. (It should be mentioned, however, that for optimal control problems, the penalty parameter for the control costs is usually assumed to be part of the problem specification and given, while its appropriate choice depending on the measurement error is of course a critical part of regularisation.) Conversely, optimal control problems can themselves be ill-posed (e.g., so-called "bang-bang" control problems), and regularisation theory can be a valuable guide for deriving robust algorithms and optimal convergence rates for approximations.

The papers collected in this special issue present topics at the current forefront of research in optimal control of differential equations and illustrate their relation to inverse problems. In the following, we put these contributions into context.

In recent years, an increasingly important focus area for research in optimal control was on non-smooth problems. Here as well, the study of non-differentiable functionals was motivated by sparse controls, which are functions whose support is of negligible Lebesgue measure; such controls are studied in [4]. A related topic is the use of the total variation as control costs, leading to piecewise constant controls [5, 6, 7, 8]. Current research is focused on optimization of variational inequalities, where the relation between control and state is itself not differentiable. Such problems are challenging also due to their non-convexity and require adapted tools for their study; see [9] in this issue.

Another emerging topic in optimal control is concerned with nonlocal models given by (time or space) fractional differential equations. Here [10] studies space-fractional elliptic partial differential equations, where the control enters into a generalized boundary condition, which for these models have to be specified on (a subset of) the exterior of the domain.

Strongly related to optimal control is the field of *shape optimisation*, where the sought-for control is not a function but the domain on which the partial differential equation is defined. Such problems can be formulated as optimal control problems either using level set methods [11], characteristic functions [12], or mappings to a reference domain [13]. In the latter case, the question of differentiability of the functional with respect to this mapping is of particular importance. Of interest are also appropriate geometric penalty terms for such functionals, where again the total variation is appealing [5, 6].

An important issue is the design of proper discretisations that respect the functional-analytic structure of optimal control problems and allow their efficient numerical solution. In the context of inverse problems, a crucial aspect here is robustness with respect to the penalty parameter, and this question is studied in [14]. A similar issue also arises in the numerical solution, and in particular the preconditioning, of the discretised linear systems, which is studied in [15]. For efficiency, a useful tool is adaptive discretisation using *a posteriori estimators*, where the numerical solution on a coarse discretisation is used to locally estimate the discretisation error and refine the discretisation accordingly, allowing to reach a desired accuracy with a comparatively small number of degrees of freedom [8]. Here again, a particular focus is on total variation, where a consistent discretisation is more challenging than for typical Sobolev-space penalties; in this issue, this is treated in [7, 8, 6].

Optimisation methods for computing optimal controls have historically mostly taken the form of classical gradient or (quasi-)Newton methods; however, recently interest has grown in methods known from inverse problems and mathematical imaging such as inertially accelerated gradient methods [16] and stochastic or Kaczmarz-type methods [11].

Regarding applications, prominent models considered in the optimal control context are related to biomedical imaging using acoustic [15, 12], elastic [17], or electromagnetic [11] waves, to optical diffusive imaging [16], or to electrical impedance tomography [8]. Another application with a long history in optimal control is data assimilation, where the sought-after control is the initial condition in a diffusive time-dependent equation. The classical context is of course weather prediction, but similar questions also arise in financial models [18]; it is also strongly related to the prototypical ill-posed problem of the backward heat equation [19]. Another challenging application tackled in [13] is of fluid-structure interactions described by a coupled system of Navier–Stokes and Lamé equations.

Finally, ill-posed optimisation problems in the form of non-coercive saddle-point problems (arising as optimality conditions of energy functionals) are treated in [17].

References

[1] Fredi Tröltzsch. *Optimal Control of Partial Differential Equations*, volume 112 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2010. Translated from the 2005 German original by Jürgen Sprekels. doi:10.1090/gsm/112.

[2] Michael Hinze, Réne Pinnau, Michael Ulbrich, and Stefan Ulbrich. *Optimization with PDE Constraints*. Springer, 2009. doi:10.1007/978-1-4020-8839-1.

[3] Juan Carlos De los Reyes. *Numerical PDE-Constrained Optimization*. Springer, 2015. doi:10.1007/978-3-319-13395-9.

[4] Eduardo Casas and Karl Kunisch. Using sparse control methods to identify sources in linear diffusion-convection equations. *Inverse Problems*, 35:114002, 2019. doi:10.1088/1361-6420/ab331c.

[5] Ronny Bergmann, Marc Herrmann, Roland Herzog, Stephan Schmidt, and José Vidal Núñez. Total variation of the normal vector field as shape prior. *Inverse Problems*, Accepted manuscript, 2020. doi:10.1088/1361-6420/ab6d5b.

[6] Ronny Bergmann, Marc Herrmann, Roland Herzog, Stephan Schmidt, and José Vidal Núñez. Discrete total variation of the normal vector field as shape prior with applications in geometric inverse problems. *Inverse Problems*, Accepted manuscript, 2020. doi:10.1088/1361-6420/ab6d5c.

[7] Michael Hinze and Tran Nhan Tam Quyen. Finite element approximation of source term identification with TV-regularization. *Inverse Problems*, 35:124004, 2019. doi:10.1088/1361-6420/ab3478.

[8] Bangti Jin and Yifeng Xu. Adaptive reconstruction for electrical impedance tomography with a piecewise constant conductivity. *Inverse Problems*, 36:014003, 2020. doi:10.1088/1361-6420/ab261e.

[9] Stanisław Migórski, Akhtar A Khan, and Shengda Zeng. Inverse problems for nonlinear quasi-hemivariational inequalities with application to mixed boundary value problems. *Inverse Problems*, 36:024006, 2020. doi:10.1088/1361-6420/ab44d7.

[10] Harbir Antil, Ratna Khatri, and Mahamadi Warma. External optimal control of nonlocal PDEs. *Inverse Problems*, 35:084003, 2019. doi:10.1088/1361-6420/ab1299.

[11] Alex J Hiles and Oliver Dorn. Sparsity and level set regularization for near-field electromagnetic imaging in 3D. *Inverse Problems*, 2020.

[12] Maïtine Bergounioux, Élie Bretin, and Yannick Privat. How to position sensors in thermo-acoustic tomography. *Inverse Problems*, 35:074003, 2019. doi:10.1088/1361-6420/ab0e4d.

[13] Johannes Haubner, Michael Ulbrich, and Stefan Ulbrich. Analysis of shape optimization problems for unsteady fluid-structure interaction. *Inverse Problems*, 36:034001, 2020. doi:10.1088/1361-6420/ab5a11.

[14] Fernando Gaspoz, Christian Kreuzer, Andreas Veeser, and Winnifried Wollner. Quasi-best approximation in optimization with PDE constraints. *Inverse Problems*, 36:014004, 2020. doi:10.1088/1361-6420/ab47f3.

[15] Alexander Beigl, Otmar Scherzer, Jarle Sogn, and Walter Zulehner. Preconditioning inverse problems for hyperbolic equations with applications to photoacoustic tomography. *Inverse Problems*, 36:014002, 2020. doi:10.1088/1361-6420/ab3d08.

[16] Rongfang Gong, Bernd Hofmann, and Ye Zhang. A new class of accelerated regularization methods, with application to bioluminescence tomography. *Inverse Problems*, Accepted Manuscript, 2020. doi:10.1088/1361-6420/ab730b.

[17] Baasansuren Jadamba, Akhtar A. Khan, Michael Richards, and Miguel Sama. A convex inversion framework for identifying parameters in saddle point problems with applications to inverse incompressible elasticity. *Inverse Problems*, 2020.

[18] Martin Burger, Jan-Frederik Pietschmann, and Marie-Therese Wolfram. Data assimilation in price formation. *Inverse Problems*, Accepted Manuscript, 2020. doi:10.1088/1361-6420/ab6d5a.

[19] Jun Liu and Mingqing Xiao. Quasi-boundary value methods for regularizing the backward parabolic equation under the optimal control framework. *Inverse Problems*, 35:124003, 2019. doi:10.1088/1361-6420/ab401e.