Calibrations in families for minimal networks

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June 1, 2018

Abstract

We define the calibrations in families for minimal Steiner networks and we explain through an example its convenience with respect to the "classical" notions of calibrations for the Steiner Problem.

The *Steiner Problem* in its classical formulation reads as follow: given a finite collection $S = \{p_1 \dots, p_m\}$ of points in the plane find a connected set with minimal length that contains S. It is well known that minimizers are finite union of segments that meet in triple junctions forming angles of 120 degrees. Finding *explicitly* a solution, it is however much more challenging (even numerically). In 1995 Brakke [3] and more recently Amato, Bellettini and Paolini [1] introduced an alternative approach to the Steiner problem rephrasing it in a *covering space* setting: minimizing the perimeter among constrained sets in a suitable covering space of $\mathbb{R}^2 \setminus S$ is equivalent to minimize the length among all connected planar networks that contain the given m points. The covering, here denoted by Y, can be constructed by a cut and paste procedure that we sketchy describe here (see [1] for details). Consider a network in the plane that contains S and another Lipschitz curve Σ that connects the points and does not intersect the network. The union of these two (the network and the curve) creates a partition in m regions of the plane. Consider m copies of $\mathbb{R}^2 \setminus S$ and lift each of these regions to one of the copies. Moreover introduce an equivalence relation that identifies the points along the curve of the different copies in such a way that the m regions can be seen as one set E in the covering space Y (given by the union of the copies together with the equivalence relation). Then the perimeter of the set in Y is twice the length of the network. The set E has some special features: it is a set of finite perimeter in Y such that for almost every x in the base space there exists exactly one point y of E such that $p(y) = x$, where p is the projection onto the base space $\mathbb{R}^2 \setminus S$. We denote by $\mathcal{P}_{constr}(Y)$ the space of all sets in Y satisfying the previous properties. Then we look for

$$\inf \{P(E) : E \in \mathcal{P}_{constr}(Y)\} . \tag{0.1}$$

We underline that this quantity is independent on the choice of the curve Σ in the construction of Y.
Our first goal in [5] was to introduce a theory of *calibrations* for Problem (0.1).

Definition 0.1. Given $E \in \mathcal{P}_{constr}(Y)$ a calibration for E is a (sufficiently regular) vector field $\Phi : Y \to \mathbb{R}^2$ such that

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i) $\mathrm{div}\Phi = 0$,

ii) $|\Phi_i - \Phi_j| \leq 2$ for every $i, j = 1, \ldots m$,

iii) $\int_Y \Phi \cdot D\chi_E = P(E)$,

where with Φ_i we denote the restriction of Φ on the i-th sheet of the covering Y.

The main purpose of searching for a calibration is to show easily the minimality of a certain candidate. Indeed we have the following:

Theorem 0.2. *If $\Phi : Y \to \mathbb{R}^2$ is a calibration for $E \in \mathcal{P}_{constr}(Y)$, then E is a minimizer among all sets in $\mathcal{P}_{constr}(Y)$.*

Finding a calibration is not easy. In [5] we exhibit a calibration only in the case S is composed of 3 and 4 points, but for general configuration of points it seems to be an hard task. In particular it is a long standing open problem to find a calibration when S is composed of points lying at the vertices of a regular polygon [4]. Indeed, even if different notions of calibrations are present in the literature (see for instance [7, 8]) and despite the effort of more than one author, this problem has never been addressed. This leads to a more basic question: does a calibration always exist?
In [6] we prove that if $\Phi : Y \to \mathbb{R}^2$ is a calibration for $E \in \mathcal{P}_{constr}$, then E is a minimizer not only among all (constrained) finite perimeter sets, but also in the larger class of finite linear combinations of characteristic functions of (constrained) finite perimeter sets. Then if there exists an element of this larger class with strictly less energy of the minimizer of Problem (0.1), a calibration for such a minimizer cannot exist. This is the case when $S = \{p_1, \ldots, p_5\}$ with p_i the vertices of a regular pentagon (see Figure 1).

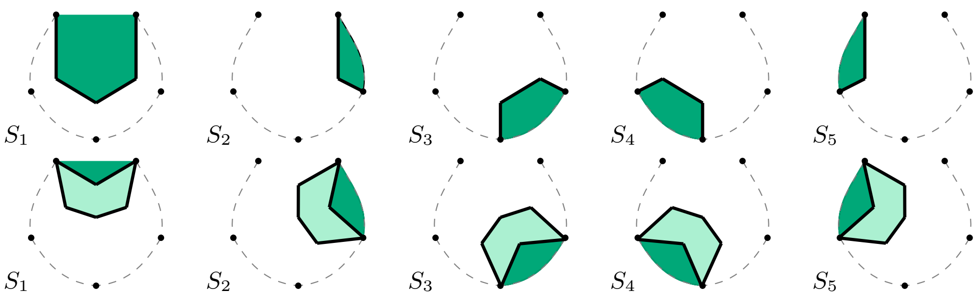

Figure 1: Let $S = \{p_1, \ldots, p_5\}$ be the set of the vertices of a regular pentagon in $\mathbb{R}^2$. We represent in the first row the set $E \in \mathcal{P}_{constr}$ minimizing (0.1) (to be precise its characteristic function u). In the second row it is depicted a BV function w, a linear combinations of characteristic functions of constrained finite perimeter sets, whose total variation is strictly less than the total variation of u. White corresponds to the value 0, light green to 1/2 and dark green to 1. This counterexample has been found by Bonafini [2] in the framework of rank one tensor valued measures and we have "translated" it in our setting.

This example clearly highlights a critical issue of the theory of calibrations. Is it then possible to slightly change the definition to overcome the problem? In [5] we introduce the notion of *calibration in families*: the strategy is to divide the set of competitors in a suitable way, defining an appropriate (and weaker) notion of calibration. Then calibrating the candidate minimizers in each family and comparing their perimeter one finds the minimizers of Problem (0.1). Thanks to this procedure in [5] we prove the minimality of the Steiner configurations spanning the vertices of a regular hexagon and pentagon.
We give now a more detailed explanation of the division in families. The competitors that belong to the same class share a property related to the projection of their essential boundary onto the base set $\mathbb{R}^2 \setminus S$. In particular we define a family as

$$\mathcal{F}(\mathcal{J}) := \{E \in \mathcal{P}_{constr}(Y) : \mathcal{H}^1(E^{i,j}) \neq 0 \text{ for every } \{i,j\} \in \mathcal{J}\},$$

where $\mathcal{J} \subset \{1,\dots,m\} \times \{1,\dots,m\}$ and $E^{i,j} := \partial^* E^i \cap \partial^* E^j$ and E^i is the restriction of E to the sheet i. In the easier case in which the set S consists of m points located on the boundary of a convex set Ω, Problem (0.1) is equivalent to a minimal partition problem. Any competitor $E \in \mathcal{P}_{constr}(Y)$ induces a partition $\{A_1,\dots,A_m\}$ of Ω where A_i are the so–called phases. Our classification in families depends on the topology of the complementary of the network as we define a family simply prescribing which phases A_i "touch" each other (see [5, Lemma 4.8] and [6] for the generalization to any configuration of the points of S). Then in each family $\mathcal{F}(\mathcal{J})$ we can weaken the notion of calibrations requiring that $|\Phi_i(x) - \Phi_j(x)| \leq 2$ for every $i,j = 1,\dots m$ such that $\{i,j\} \in \mathcal{J}$. If there exists a calibration Φ_k for E_k in $\mathcal{F}(\mathcal{J}^k)$, then E_k is a minimizer in $\mathcal{F}(\mathcal{J}^k)$. Suppose moreover that there exist $\mathcal{J}_1,\dots,\mathcal{J}_N$ such that $\mathcal{P}_{constr} = \bigcup_{k=1}^N \mathcal{F}(\mathcal{J}^k)$ then the solution of Problem (0.1) is the E_k with less energy among the minimizers in all different $\mathcal{F}(\mathcal{J}^k)$.
What is the advantage of calibrations in families? We consider again as example $S = \{p_1,\dots,p_5\}$ with p_i the vertices of a regular pentagon. In this case we cover the whole space $\mathcal{P}_{constr}(Y)$ with five families (in particular we get a family requiring that the phase A_1 touches A_3 and A_4 and we obtain the other four families with a cyclical permutation of the indices). We are then able to find a calibration in each of this family (see Figure 2). Beyond this single successful example, in the general case when the division in families is the finest possible one, it classifies the competitors relying on their topological type. Moreover as the minimizer is unique in each family by the convexity of the distance within any combinatorial type, it is possible to prove that a calibration in such a family always exists.

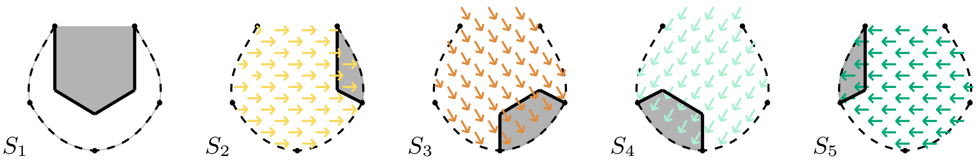

Figure 2: A calibration for the five vertices of a regular pentagon in a fixed family.

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