Tensor-Free Algorithms for Convex Liftings of Bilinear Inverse Problems with Applications to Masked Phase Retrieval

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***Abstract*—The masked phase retrieval problem is a challenging inverse problem that occurs in many applications. It can be regarded as a quadratic, or, more generally, bilinear inverse problem. Using a convexification based on tensorial lifting and first-order proximal algorithms, it may be solved by singular-value thresholding. However, computations on the tensor product are often impracticable. To overcome this limitation, we propose tensor-free versions of singular-valued thresholding methods that base on low-rank representations and an augmented Lanczos process. We moreover demonstrate that a novel reweighting technique can improve the methods' convergence behavior and rank evolution.**

I. Introduction

The classical phase retrieval problem consists in the recovery of a possibly complex-valued signal u from the magnitudes $|\mathcal{F}[u]|$ of its Fourier transform. Problems of this kind occur in crystallography [1], [2], astronomy [3], [4], and laser optics [5], [6]. To be more precise, we consider the two-dimensional masked Fourier phase retrieval problem, where we aim at recovering an image $u \in \mathbf{C}^{N\times N}$ from the masked Fourier intensities $|\mathcal{F}_M[d_\ell \odot u]|$ for $\ell = 0, \dots, L-1$. Here, the unknown signal u is first pointwise multiplied with a mask $d_\ell \in \mathbf{C}^{N\times N}$ and afterwards M-point Fourier transformed by

$$(\mathcal{F}_M[v])[m_2, m_1] := \sum_{n_2,n_1=0}^{N-1} v[n_2, n_1] e^{-2\pi i (n_2 m_2 + n_1 m_1)/M}.$$

Taking squares, the masked Fourier phase retrieval problem $|\mathcal{F}[u]|^2 = g^\dagger$ can be interpreted as a quadratic inverse problem which is, requiring $u = v$, a special case of the bilinear inverse problem

$$K(u, v) = g^\dagger, \tag{1}$$

where K is a bilinear operator from $\mathcal{H}_1 \times \mathcal{H}_2$ into $\mathcal{K}$, and where $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{K}$ denote finite-dimensional real Hilbert spaces.

II. First-order algorithms for convex liftings

Inspired by the well-known PhaseLift algorithm [7], [8], we tackle problem (1) by tensorial lifting [9]. Based on the universal property of the tensor product, we can always find a unique linear $\breve{K} : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{K}$ with $\breve{K}(u \otimes v) = K(u, v)$ such that (1) is equivalent to

$$\breve{K}(w) = g^\dagger \qquad \text{with} \qquad \operatorname{rank}(w) = 1. \tag{2}$$

To find the solution in the noise-free case, we minimize the nuclear norm $||\cdot||_{\mathcal{H}_1 \otimes_\pi \mathcal{H}_2}$ over the solution set of the lifted problem, i.e. we solve the following relaxed convex version of (2):

$$\text{minimize} \quad ||w||_{\mathcal{H}_1 \otimes_\pi \mathcal{H}_2} \quad \text{with} \quad \breve{K}(w) = g^\dagger. \tag{3}$$

Minimizers for (3) may be determined by first-order proximal algorithms, for instance, the primal-dual method in [10]:

$$y^{(n+1)} := y^{(n)} + \sigma(\breve{K}(w^{(n)} + \theta(w^{(n)} - w^{(n-1)})) - g^\dagger), \tag{4}$$
$$w^{(n+1)} := \mathcal{S}_{\tau\alpha}(w^{(n)} - \tau \breve{K}^*(y^{(n+1)})), \tag{5}$$

for suitable parameters σ, τ, α, and θ. This and other proximal methods incorporate $\mathcal{S}_{\tau\alpha}$ which is the singular-value (soft) thresholding with respect to $\mathcal{H}_1$ and $\mathcal{H}_2$. Note that adaptations to quadratic problems as well as to the case of noisy data are straightforward. Also, other first-order algorithms such as ADMM [11] may be used.

III. Tensor-free computation and reweighting

The main drawback of algorithm (4–5) are the computations in the usually high-dimensional tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$. To overcome this issue, we exploit that singular-value thresholding algorithms enforce low-rank iterates. By storing the singular values/vectors instead of the full tensor, we can efficiently compute (4) using the universal property. For the update (5), we apply a restarted augmented Lanczos process [12] to compute the singular value decomposition embracing all singular values greater than $\tau\alpha$. This process does not require the application of $\breve{K}^*$ but rather the action of the adjoints of the linear mappings $u \mapsto K(u, v')$, $v \mapsto K(u', v)$ for fixed $u' \in \mathcal{H}_1$, $v' \in \mathcal{H}_2$. In total, the algorithm can be performed in a tensor-free way.

For further rank reduction, we propose a novel reweighting heuristic for $\mathcal{H}_1$ and, analogously, for $\mathcal{H}_2$ by replacing the inner product by

$$\langle u, v\rangle_{\tilde{\mathcal{H}}_1} := \langle u, v\rangle_{\mathcal{H}_1} - \sum_{n=0}^{R-1} \lambda_n \langle u, e_n\rangle_{\mathcal{H}_1} \langle e_n, v\rangle_{\mathcal{H}_1}$$

with $\lambda_n \in (0, 1)$ and orthonormal vectors e_n. Choosing e_n as the leading singular vectors of $w^{(n)}$, we can promote these directions in the subsequent singular value decompositions that occur in (5).

IV. Application to masked phase retrieval

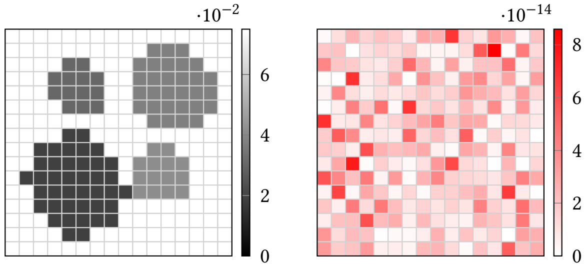
We now apply the developed algorithm to phase retrieval with randomly generated Rademacher masks [13], [14]. A first reconstruction of a synthetic image is shown in Fig. 1. In Fig. 2–3, we can observe the positive effects of the Lanczos process and the reweighting. By equipping $\mathcal{H}_1 = \mathcal{H}_2 = \mathbf{C}^{N_2 \times N_1}$ with a Sobolev norm, we can even recover pixels that are not covered by the generated masks, see Fig. 4.

V. Conclusion

Exploiting the low-rank representations of $w^{(n)}$ together with the augmented Lanczos process and reweighting technique, we obtain an efficient lifting algorithm that avoids direct computations on the tensor product. Moreover, by choosing the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ suitably, we can directly incorporate a priori assumptions like smoothness properties into the nuclear norm. The proposed algorithm cannot only be used for phase retrieval but also for further bilinear or quadratic recovery problems like deautoconvolution, blind deconvolution, and parallel magnetic resonance imaging.

Acknowledgment

We gratefully acknowledge the funding of this work by the Austrian Science Fund (FWF) within project P 28858.



(a) Recovered signal (b) Absolute error

Fig. 1. Masked phase retrieval based on the proposed primal-dual algorithm with augmented Lanczos bidiagonalization and with Hilbert space reweighting. The algorithm is terminated after 1 000 iterations, the reweighting is repeated every 10 iterations.

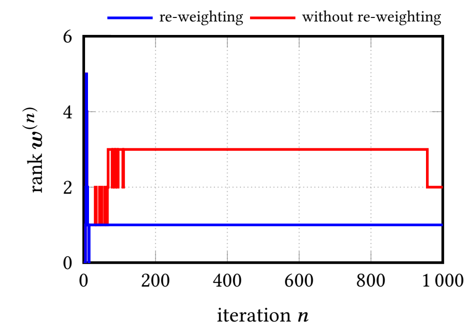


Fig. 2. Evolution of the rank during the proposed primal-dual algorithm. The Hilbert space reweighting additionally reduces the rank of the iteration.

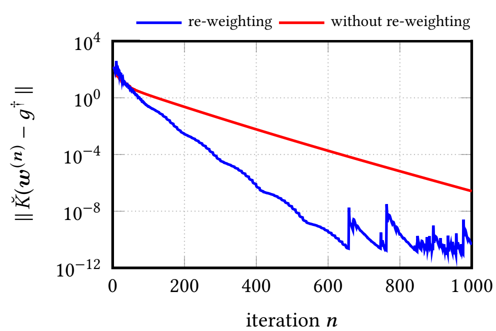
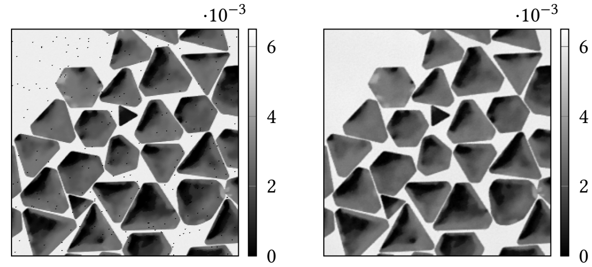


Fig. 3. Evolution of the data fidelity term. The Hilbert space reweighting increases the convergence rate. Due to numerical issues the convergence stagnates at a data fidelity of $\approx 10^{-10}$.



(a) Phase retrieval based on L^2-spaces. (b) Phase retrieval based on Sobolev spaces.

Fig. 4. Comparison of the reconstruction with respect to different Hilbert space norms. For discrete L^2-norms, image defects (isolated black pixels) result from pixels that are not covered by the employed masks. Due to the a priori smoothing, these defects are corrected when using discrete Sobolev norms. The test image has been created based on a transmission electron microscopy image in [15] published open access under a Creative Commons Attribution 3.0 Unported Licence.

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