Ambiguities in one-dimensional phase retrieval from Fourier magnitudes

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Phase retrieval problem
**Formulation of the problem**

**Problem (Phase retrieval)**

*Recover the unknown complex-valued signal*

\[ x := (x[n])_{n \in \mathbb{Z}} \]

*with finite support from the Fourier intensity*

\[ |\hat{x}(\omega)| \quad (\omega \in \mathbb{R}). \]

**Definition (Discrete-time Fourier transform)**

\[ \hat{x}(\omega) := \sum_{n \in \mathbb{Z}} x[n] e^{-i\omega n} \quad (\omega \in \mathbb{R}) \]
Trivial ambiguities

Example

Let $x$ be a complex-valued signal. Then

- the rotated signal
  \[(y[n]) := (e^{i\alpha} x[n]),\]

- the shifted signal
  \[(y[n]) := (x[n - n_0]),\]

- the reflected, conjugated signal
  \[(y[n]) := (\overline{x[-n]}),\]

have the same \textsc{Fourier} intensity $|\hat{x}|$. 
Non-trivial ambiguities

Example

**Frequency domain**

\[ |\hat{x}(\omega)| \]

**Time domain**

\[ x[n] \]

Absolute value: \(|x[n]|\)

Phase: \(\text{arg}(x[n])\)

Polar representation: \(x[n]\)
Characterizing the solution set
### Autocorrelation signal and function

#### Definition (Autocorrelation signal)

\[
a[n] := \sum_{k \in \mathbb{Z}} x[k] x[k + n] \quad (n \in \mathbb{Z}).
\]

- The autocorrelation signal is **conjugate symmetric**, i.e.

\[
\overline{a[-n]} = \sum_{k \in \mathbb{Z}} x[k] x[k - n] = \sum_{k \in \mathbb{Z}} x[k + n] x[k] = a[n] \quad (n \in \mathbb{Z}).
\]

#### Definition (Autocorrelation function)

\[
A(\omega) := \sum_{n \in \mathbb{Z}} a[n] e^{-i\omega n} = \sum_{n=-N+1}^{N-1} a[n] e^{-i\omega n}.
\]

- The autocorrelation function is a **non-negative trigonometric polynomial** of degree \(N - 1\).
• Relationship to the **FOURIER** transform:

\[
|\hat{x}(\omega)|^2 = \left( \sum_{n \in \mathbb{Z}} x[n] e^{-i\omega n} \right) \left( \sum_{k \in \mathbb{Z}} \overline{x[k]} e^{i\omega k} \right) = \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} x[k + n] \overline{x[k]} e^{-i\omega n} = A(\omega).
\]

**Equivalent problem**

Find a trigonometric polynomial \(B\) such that

\[
|B(\omega)|^2 = A(\omega).
\]
**Definition (associate polynomial)**

\[
P_A(e^{-i\omega}) = e^{-i\omega(N-1)} A(\omega)
\]

- The algebraic polynomial \( P_A \) is thus defined by

\[
P_A(z) := \sum_{n=0}^{2N-2} a[n - N + 1] z^n \quad \text{with} \quad a[-n] = \overline{a[n]}.
\]

- Obviously, we have

\[
A(\omega) = \left| P_A(e^{-i\omega}) \right|.
\]

- \( P_A \) has the factorization

\[
P_A(z) = a[N - 1] \prod_{j=1}^{N-1} (z - \gamma_j)(z - \overline{\gamma_j}^{-1}).
\]
Factorization of the autocorrelation function

• For \( z := e^{-i\omega} \), the absolute value of the linear factors is

\[
\left| \left( e^{-i\omega} - \gamma_j \right) \left( e^{-i\omega} - \gamma_j^{-1} \right) \right| = \left| e^{-i\omega} - \gamma_j \right| \left| \gamma_j^{-1} \right| \left| \gamma_j - e^{i\omega} \right|
\]

\[
= \left| \gamma_j \right|^{-1} \left| e^{-i\omega} - \gamma_j \right|^2.
\]

• \( A \) has the factorization

\[
A(\omega) = \left| P_A(e^{-i\omega}) \right| = \left| a[N - 1] \right| \prod_{j=1}^{N-1} \left| \left( e^{-i\omega} - \gamma_j \right) \left( e^{-i\omega} - \gamma_j^{-1} \right) \right|
\]

\[
= \left| a[N - 1] \right| \prod_{j=1}^{N-1} \left| \beta_j \right|^{-1} \prod_{j=1}^{N-1} \left| e^{-i\omega} - \beta_j \right|^2 = \left| B(\omega) \right|^2
\]

with \( \beta_j \in (\gamma_j, \gamma_j^{-1}) \).
Theorem (Beinert, Plonka [2015])

Let $A$ be a non-negative trigonometric polynomial. Then the problem

$$|B(\omega)|^2 = A(\omega)$$

has at least one solution. Every solution has a representation of the form

$$B(\omega) = e^{i\alpha + i\omega n_0} \sqrt{|a[N - 1]| \prod_{j=1}^{N-1} |\beta_j|^{-1} \cdot \prod_{j=1}^{N-1} (e^{-i\omega} - \beta_j)},$$

where $\beta_j$ can be chosen from the zero pair $(\gamma_j, \bar{\gamma}_j^{-1})$ of the associated polynomial $P_A$. 
Representation of the ambiguities in time domain

**Definition (Convolution of signals)**

\[(x_1 * x_2)[n] := \sum_{k \in \mathbb{Z}} x_1[k] x_2[n - k].\]

**Theorem (Beinert, Plonka [2015])**

Let \(x\) be a signal with finite support and factorization

\[x = x_1 * x_2.\]

Then the signal

\[y := e^{i\alpha} \left( x_1[\cdot - \cdot] \right) * (x_2[\cdot - n_0])\]

has the same \textit{Fourier intensity} \(|\hat{x}|\) and \textit{all signals with the Fourier intensity} \(|\hat{x}|\) can be represented in this manner.
Ensuring uniqueness
Phase retrieval of non-negative signals

Example

Unique non-negative solution

Full non-negative solution set

Theorem (Beinert [2015])

The sets of non-negative signals with support length $N > 3$ that

- can be recovered uniquely up to reflection
- cannot be recovered uniquely up to reflection

from their Fourier intensities are unbounded sets of infinite Lebesgue measure.
Knowledge of additional moduli

- Recover $x$ from $|\hat{x}|$ and $|x[N-1-\ell]|$ for an $\ell$.
- Assume that there exist two non-trivial solutions $x$ and $\tilde{x}$.
- For $|x[N-1-\ell]| = |\tilde{x}[N-1-\ell]|$, Vieta’s formulae yield the condition

$$\prod_{j=1}^{N-1} |\beta_j|^{-\frac{1}{2}} \cdot \left| \sum_{1 \leq k_1 < \cdots < k_\ell \leq N-1} \beta_{k_1} \cdots \beta_{k_\ell} \right| = \prod_{j=1}^{N-1} |\tilde{\beta}_j|^{-\frac{1}{2}} \cdot \left| \sum_{1 \leq k_1 < \cdots < k_\ell \leq N-1} \tilde{\beta}_{k_1} \cdots \tilde{\beta}_{k_\ell} \right|.$$

Theorem (Beinert, Plonka [2015])

Almost every signal $x$ can be recovered from $|\hat{x}|$ and $|x[N-1-\ell]|$ for an arbitrary $\ell \neq (N-1)/2$ up to rotations, for $\ell = (N-1)/2$ up to reflection/conjugation and rotation.
Knowledge of additional moduli

Example

**Fourier intensity: \( |\hat{x}(\omega)| \)**

**Time domain**

| Absolute value: \( |x[n]| \) |
| --- |
| 0 | 0.5 | 1.0 | 1.5 |
| -1 | 0 | 1 | 2 | 3 | 4 |

**Phase: \( \arg(x[n]) \)**

<table>
<thead>
<tr>
<th>Phase: ( \arg(x[n]) )</th>
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<tbody>
<tr>
<td>-( \pi )</td>
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<tr>
<td>-1</td>
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**Polar representation: \( x[n] \)**
Knowledge of additional phases

Theorem (Beinert [2015])

Let $\ell_1$ and $\ell_2$ two different integers in $\{0, \ldots, N-1\}$. Then almost every signal $x$ can be uniquely recovered from $|\hat{x}|$ and the phases

$$\arg x[N-1-\ell_1] \quad \text{and} \quad \arg x[N-1-\ell_2] \quad (\ell_1 + \ell_2 \neq N-1).$$

For $\ell_1 + \ell_2 = N-1$, the recovery of the unknown signal is only unique up to reflection/conjugation, except for the case where the phase of both end points is given.
Interference with reference signal

Real case: Kim, Hayes [1993]
Complex case: Raz, Dudovich, Nadler [2013]

Theorem (Beinert, Plonka [2015])

Let \( x \) and \( h \) be complex-valued signals with finite support, and assume that the factorization of their symbols

\[
\hat{x}(\omega) = e^{i\omega n_1} x[N_1 - 1] \prod_{j=1}^{N_1-1} (e^{-i\omega} - \eta_j)
\]

and

\[
\hat{h}(\omega) = e^{i\omega n_2} h[N_2 - 1] \prod_{j=1}^{N_2-1} (e^{-i\omega} - \gamma_j)
\]

have no common zeros. Then \( x \) and \( h \) can be uniquely recovered from \(|\hat{x}(\omega)|, |\hat{h}(\omega)|\) and \(|\hat{x}(\omega) + \hat{h}(\omega)|\) up to common trivial ambiguities.
Sketch of proof

- Assume there are two solutions $x[n]$, $h[n]$ and $\tilde{x}[n]$, $\tilde{h}[n]$.

- Use the factorization in the frequency domain:
  \[
  \hat{x}(\omega) = e^{i\omega n_1} \hat{x}_1(\omega) \hat{x}_2(\omega) \quad \text{and} \quad \hat{x}(\omega) = e^{i\alpha_1} e^{i\omega k_1} \hat{x}_1(\omega) \hat{x}_2(\omega),
  \]
  \[
  \hat{h}(\omega) = e^{i\omega n_2} \hat{h}_1(\omega) \hat{h}_2(\omega) \quad \text{and} \quad \hat{h}(\omega) = e^{i\alpha_2} e^{i\omega k_2} \hat{h}_1(\omega) \hat{h}_2(\omega).
  \]

- Consider the identity
  \[
  \left| \hat{x}(\omega) + \hat{h}(\omega) \right|^2 = \left| \hat{x}(\omega) + \hat{h}(\omega) \right|^2.
  \]
Interference with reference signal

Theorem (Beinert [2015])

Let $f$ and $h$ be complex-valued continuous-time signals with compact support, and assume that the factorization of their Laplace transforms

$$F(\zeta) = C_1 \zeta^{m_1} \, e^{\zeta \gamma_1} \prod_{j=1}^{\infty} \left(1 - \frac{\zeta}{\xi_j}\right) e^{\frac{\zeta}{\xi_j}}$$

and

$$H(\zeta) = C_2 \zeta^{m_2} \, e^{\zeta \gamma_2} \prod_{j=1}^{\infty} \left(1 - \frac{\zeta}{\eta_j}\right) e^{\frac{\zeta}{\eta_j}}$$

have no common zeros. Then $f$ and $h$ can be uniquely recovered from $|\hat{f}(\omega)|$, $|\hat{h}(\omega)|$ and $|\hat{f}(\omega) + \hat{h}(\omega)|$ up to common trivial ambiguities.
Summary/Outlook

- Characterization of the ambiguities in the one-dimensional discrete-time phase retrieval problem.
- Investigation of the quality of different a priori conditions and additional data.

- Phase retrieval in higher dimensions.
- Transferring further results between the discrete-time and continuous-time problem.
- Investigation and development of numerical algorithms.
Thank you for the attention.

Appendix


