Ambiguities in one-dimensional phase retrieval from Fourier magnitudes

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based on joint work with
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Mathematical Signal Processing and Phase Retrieval
2 September 2014
Formulation of the problem

Problem (Phase retrieval)

*Recover the unknown complex-valued signal*

\[
x := (x[n])_{n \in \mathbb{Z}}
\]

*with finite support from its Fourier intensities*

\[
|\hat{x}(\omega)|^2.
\]

Definition (Discrete Fourier transform)

\[
\hat{x}(\omega) := \sum_{n \in \mathbb{Z}} x[n] e^{-i\omega n} \quad (\omega \in [-\pi, \pi)).
\]
Trivial ambiguities

**Example**

Let $x$ be a complex-valued signal. Then

- the **rotated** signal
  
  $$ (y[n]) := (e^{i\alpha} x[n]) , $$

- the **shifted** signal
  
  $$ (y[n]) := (x[n - n_0]) , $$

- the **reflected conjugated** signal
  
  $$ (y[n]) := (\overline{x[-n]}) $$

have the same **FOURIER** intensities $\left| \hat{x}(\omega) \right|^2$. 
Nontrivial ambiguities

Example

**Example**

![FFT Plot](image)

**Modulus:** $|x[n]|$

**Phase:** $\text{arg}(x[n])$

**Polar plot of** $x$

**Fourier intensities:** $|\hat{x}(\omega)|$
Relation to the autocorrelation

- **Autocorrelation signal**
  \[ a[n] := \sum_{k \in \mathbb{Z}} x[k] \overline{x[k+n]} \quad (n \in \mathbb{Z}). \]

- **Autocorrelation function**
  \[ A(\omega) := |\hat{x}(\omega)|^2 = \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} x[n] \overline{x[k]} e^{-i\omega(n-k)} = \sum_{n \in \mathbb{Z}} a[n] e^{-i\omega n}. \]

**Equivalent problem**

Find the trigonometric polynomials \( B(\omega) \) which satisfy
\[ |B(\omega)|^2 = A(\omega). \]
Consider the complex polynomial $P_A(z)$ defined by

$$P_A(e^{-i\omega}) = e^{-i\omega(N-1)} A(\omega)$$

for an autocorrelation function with $a[-n] = \overline{a[n]}$, i.e. with $z = e^{-i\omega}$

$$P_A(z) := a[0] z^{N-1} + \sum_{n=1}^{N-1} \overline{a[n]} z^{N-1-n} + \sum_{n=1}^{N-1} a[n] z^{N-1+n}.$$

Let $\gamma_j$ be the roots of $P_A$. 

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**Associated polynomial**
Factorization

- The roots occur in pairs \((\gamma_j, \bar{\gamma}_j^{-1})\):

\[
\bar{\gamma}_j^{2N-2} P_A \left( \bar{\gamma}_j^{-1} \right)
= \bar{\gamma}_j^{2N-2} \left[ a[0] \bar{\gamma}_j^{-N+1} + \sum_{n=1}^{N-1} a[n] \bar{\gamma}_j^{-N+1+n} + \sum_{n=1}^{N-1} a[n] \bar{\gamma}_j^{-N+1-n} \right]

= a[0] \bar{\gamma}_j^{N-1} + \sum_{n=1}^{N-1} a[n] \bar{\gamma}_j^{N-1+n} + \sum_{n=1}^{N-1} a[n] \bar{\gamma}_j^{N-1-n}

= P_A (\gamma_j).

- \(P_A\) has the factorization

\[
P_A (z) = a[N-1] \prod_{j=1}^{N-1} (z - \gamma_j) \left( z - \bar{\gamma}_j^{-1} \right).
\]
Investigation of the autocorrelation
(Real case: DAUBECHIES [1992])

- Observe that

\[
\left| (e^{-i\omega} - \gamma_j) (e^{-i\omega} - \bar{\gamma}_j^{-1}) \right| = |\gamma_j|^{-1} |e^{-i\omega} - \gamma_j|^2.
\]

- \( A \) has the factorization

\[
A(\omega) = \left| P_A(e^{-i\omega}) \right| \\
= \left| a [N - 1] \prod_{j=1}^{N-1} |\gamma_j|^{-1} \prod_{j=1}^{N-1} (e^{-i\omega} - \gamma_j) \right|^2 \\
= \left| B(\omega) \right|^2.
\]
Investigation of the autocorrelation

On the other side, \( B(\omega) \) is of the form

\[
B(\omega) = e^{i\omega n_0} \sum_{n=0}^{N-1} b[n] e^{-i\omega n} = e^{i\omega n_0} b[N - 1] \prod_{j=1}^{N-1} \left( e^{-i\omega} - \beta_j \right).
\]

\( P_A \) can also be written as

\[
P_A\left( e^{-i\omega} \right) = e^{-i\omega(N-1)} A(\omega) = e^{-i\omega(N-1)} |B(\omega)|^2
\]

\[
= |b[N - 1]|^2 \prod_{j=1}^{N-1} \beta_j \prod_{j=1}^{N-1} \left( e^{-i\omega} - \beta_j \right)\left( e^{-i\omega} - \overline{\beta_j}^{-1} \right).
\]
Nontrivial ambiguities
(Real case: Bruck, Sodin [1979])

Theorem

Let $A(\omega)$ be a nonnegative trigonometric polynomial with $a[-n] = \overline{a[n]}$. Then the problem

$$|B(\omega)|^2 = A(\omega)$$

has at least one solution. Furthermore, each solution can be written in the form

$$B(\omega) = e^{i\alpha + i\omega n_0} \left[|a[N-1]| \prod_{j=1}^{N-1} |\beta_j|^{-1}\right]^{\frac{1}{2}} \prod_{j=1}^{N-1} \left(e^{-i\omega} - \beta_j\right).$$

Assuming that $A(\omega) = e^{i\omega(N-1)} P_A(e^{-i\omega})$, where $P_A(z)$ is a polynomial with paired zeros $\gamma_j$ and $\overline{\gamma_j}^{-1}$, we can choose $\beta_j \in \{\gamma_j, \overline{\gamma_j}^{-1}\}$.
Convolution

Definition (Convolution of signals)

\((x_1 * x_2)[n] := \sum_{k \in \mathbb{Z}} x_1[k] x_2[n-k].\)

Theorem (BEINERT, PLONKA-HOCH [2014])

Let \(\left|\hat{x}(\omega)\right|^2\) be given. Further, let \(x_1\) and \(x_2\) be two finite signals with \(x = x_1 * x_2\).

Then

\[y := e^{i\alpha} (x_1[-\cdot]) * (x_2[\cdot - n_0])\]

has the same FOURIER intensities. Moreover, all signals with the FOURIER intensities \(\left|\hat{x}(\omega)\right|^2\) have such a representation.
Positivity of the real signal

Example

Intensities $|\hat{\chi}(\omega)|$

Recovered signals

Full solution set

Unique solution

No nonnegative solution

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Additional end points

(Real case: Xu, Yan, Chang [1987])

- Assume that the end point of the signal

\[ x[N - 1] = e^{i\alpha} \left| a[N - 1] \prod_{j=1}^{N-1} |\beta_j|^{-1} \right|^{\frac{1}{2}} \]

is additionally given.
- Non-uniqueness: Replacing the first \( L \) zeros yields the condition

\[ \prod_{j=1}^{L} |\beta_j|^2 - 1 = 0. \]

**Theorem**

*Almost every signal* \( x \) *can be uniquely recovered from its Fourier intensities* \( |\hat{x}(\omega)|^2 \) *and its end point* \( x[N - 1] \).
Given moduli of the signal

\[ \text{(Seifert, Stolz, Donatelli, Langemann, Tasche [2006]; Langemann, Tasche [2008])} \]

- Assume that the modulus of the signal \( |x[n]|^2 \) is additionally given.
- Non-uniqueness: Again we get for \( x[N - 1] \) the condition

\[
\prod_{j=1}^{L} |\beta_j|^2 - 1 = 0.
\]

**Theorem**

*Almost every signal \( \mathbf{x} \) can be uniquely recovered from the Fourier intensities \( |\hat{x}(\omega)|^2 \) and the moduli \( |x[n]|^2 \) up to rotations.*
Interference with a known signal

(Real Case: Kim, Hayes [1990])

- Let $h$ be a known finite length signal.
- Assume that the Fourier intensities $|\hat{y}(\omega)|^2$ of
  
  $$y[n] = x[n] + h[n] \quad (n \in \mathbb{Z})$$

  are also given.
- In some cases, e.g. if $h$ is a Dirac signal or has linear phase, the phase retrieval problem is uniquely solvable.
Interference with an unknown signal

(Real case: KIM, HAYES [1993]; Complex case: RAZ, DUDOVICH, NADLER [2013])

Theorem

Let $x$ and $h$ be two complex signals with finite support and assume that the factorizations of their symbols

$$\hat{x}(\omega) = e^{i\omega n_1} x [N_1 - 1] \prod_{j=1}^{N_1-1} \left( e^{-i\omega} - \eta_j \right),$$

$$\hat{h}(\omega) = e^{i\omega n_2} h [N_2 - 1] \prod_{j=1}^{N_2-1} \left( e^{-i\omega} - \gamma_j \right),$$

have no common nonzero roots. Then $x$ and $h$ can be uniquely recovered from $|\hat{x}(\omega)|^2$, $|\hat{h}(\omega)|^2$ and $|\hat{x}(\omega) + \hat{h}(\omega)|^2$ up to trivial ambiguities.
Interference with an unknown signal

Proof.

- Assume the problem has two solutions $x[n]$, $h[n]$ and $\tilde{x}[n]$, $\tilde{h}[n]$.
- Use the factorizations
  
  $\hat{x}(\omega) = e^{i\omega n_1} \hat{x}_1(\omega) \hat{x}_2(\omega)$ and $\hat{\tilde{x}}(\omega) = e^{i\alpha_1} e^{i\omega_1 k_1} \hat{x}_1(\omega) \hat{x}_2(\omega)$,
  
  $\hat{h}(\omega) = e^{i\omega n_2} \hat{h}_1(\omega) \hat{h}_2(\omega)$ and $\hat{\tilde{h}}(\omega) = e^{i\alpha_2} e^{i\omega_2 k_2} \hat{h}_1(\omega) \hat{h}_2(\omega)$.

- Consider the identity
  
  $|\hat{x}(\omega) + \hat{h}(\omega)|^2 = |\hat{\tilde{x}}(\omega) + \hat{\tilde{h}}(\omega)|^2$. 

□
Summary/Outlook

- Characterization of the ambiguities in the one-dimensional phase retrieval problem.
- Quality of preconditions and additional data.
- Phase retrieval in higher dimension.
- Ambiguities in the Fresnel regime.

Thank you for your attention.
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Thank you for your attention.