# Ambiguities in one-dimensional phase retrieval from FOURIER magnitudes

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based on joint work with

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# Formulation of the problem

**Problem (Phase retrieval)**

*Recover the unknown complex-valued signal*

$$\mathbf{x} := (x[n])_{n\in\mathbb{Z}}$$

*with finite support from its* FOURIER *intensities*

$$\left|\widehat{x}(\omega)\right|^2 .$$

**Definition (Discrete FOURIER transform)**

$$\widehat{x}(\omega) := \sum_{n\in\mathbb{Z}} x[n]\, \mathrm{e}^{-\mathrm{i}\omega n} \qquad (\omega \in [-\pi, \pi)) .$$

# Trivial ambiguities

## Example

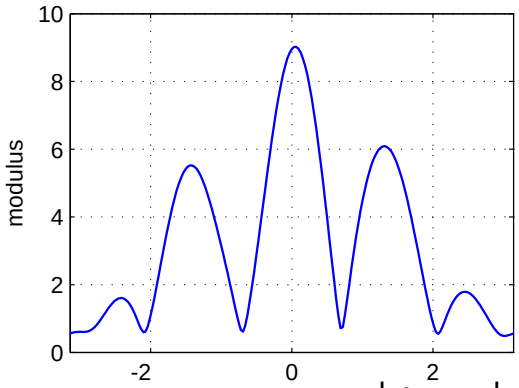
Let **x** be a complex-valued signal. Then

- the rotated signal
$$(y[n]) := \left(e^{i\alpha} x[n]\right),$$
- the shifted signal
$$(y[n]) := (x[n - n_0]),$$
- the reflected conjugated signal
$$(y[n]) := \left(\overline{x[-n]}\right)$$

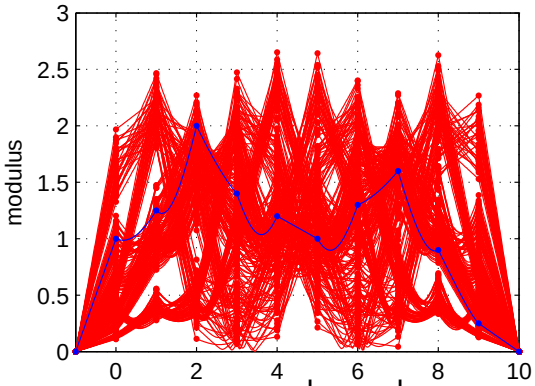
have the same FOURIER intensities $\left|\widehat{x}(\omega)\right|^2$.

# Nontrivial ambiguities

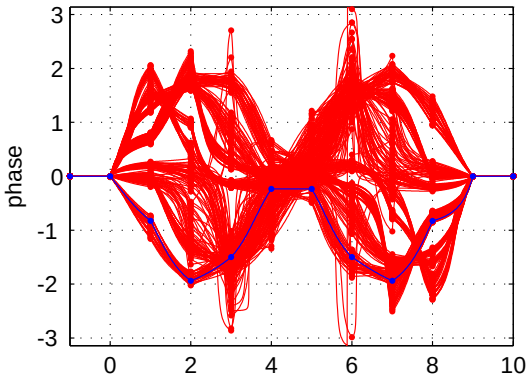
## Example



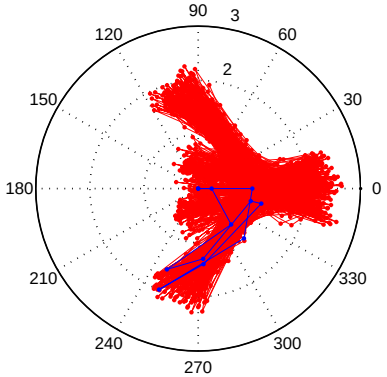

FOURIER intensities: $\left|\widehat{x}(\omega)\right|$




Modulus: $\left|x[n]\right|$




Phase: $\arg(x[n])$




Polar plot of **x**

# Relation to the autocorrelation

- Autocorrelation signal

$$a[n] := \sum_{k\in\mathbb{Z}} x[k]\,\overline{x[k+n]} \qquad (n \in \mathbb{Z}).$$

- Autocorrelation function

$$A(\omega) := \left|\widehat{x}(\omega)\right|^2 = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x[n]\,\overline{x[k]}\,\mathrm{e}^{-\mathrm{i}\omega(n-k)} = \sum_{n\in\mathbb{Z}} a[n]\,\mathrm{e}^{-\mathrm{i}\omega n}.$$

**Equivalent problem**

Find the trigonometric polynomials $B(\omega)$ which satisfy

$$\left|B(\omega)\right|^2 = A(\omega).$$

# Associated polynomial

- Consider the complex polynomial $P_A(z)$ defined by

$$P_A\left(\mathrm{e}^{-\mathrm{i}\omega}\right) = \mathrm{e}^{-\mathrm{i}\omega(N-1)} A(\omega)$$

for an autocorrelation function with $a[-n] = \overline{a[n]}$,
i. e. with $z = \mathrm{e}^{-\mathrm{i}\omega}$

$$P_A(z) := a[0]\, z^{N-1} + \sum_{n=1}^{N-1} \overline{a[n]}\, z^{N-1-n} + \sum_{n=1}^{N-1} a[n]\, z^{N-1+n}.$$

- Let $\gamma_j$ be the roots of $P_A$.

# Factorization

- The roots occur in pairs $\left(\gamma_j, \overline{\gamma}_j^{-1}\right)$:

$$\begin{aligned}
&\overline{\gamma}_j^{2N-2} P_A\left(\overline{\gamma}_j^{-1}\right) \\
&= \overline{\gamma}_j^{2N-2}\left[a[0]\,\overline{\gamma}_j^{-N+1} + \sum_{n=1}^{N-1} \overline{a[n]}\,\overline{\gamma}_j^{-N+1+n} + \sum_{n=1}^{N-1} a[n]\,\overline{\gamma}_j^{-N+1-n}\right] \\
&= \overline{a[0]}\,\overline{\gamma}_j^{N-1} + \sum_{n=1}^{N-1} \overline{a[n]}\,\overline{\gamma}_j^{N-1+n} + \sum_{n=1}^{N-1} a[n]\,\overline{\gamma}_j^{N-1-n} \\
&= \overline{P_A(\gamma_j)}.
\end{aligned}$$

- $P_A$ has the factorization

$$P_A(z) = a[N-1] \prod_{j=1}^{N-1} (z - \gamma_j)\left(z - \overline{\gamma}_j^{-1}\right).$$

# Investigation of the autocorrelation

(Real case: DAUBECHIES [1992])

- Observe that

$$\left|\left(\mathrm{e}^{-\mathrm{i}\omega}-\gamma_j\right)\left(\mathrm{e}^{-\mathrm{i}\omega}-\overline{\gamma}_j^{-1}\right)\right| = \left|\gamma_j\right|^{-1}\left|\mathrm{e}^{-\mathrm{i}\omega}-\gamma_j\right|^2 .$$

- $A$ has the factorization

$$\begin{aligned} A(\omega) &= \left|P_A\left(\mathrm{e}^{-\mathrm{i}\omega}\right)\right| \\ &= \left|a[N-1]\right| \prod_{j=1}^{N-1}\left|\gamma_j\right|^{-1}\left|\prod_{j=1}^{N-1}\left(\mathrm{e}^{-\mathrm{i}\omega}-\gamma_j\right)\right|^2 \\ &= \left|B(\omega)\right|^2 . \end{aligned}$$

# Investigation of the autocorrelation

- On the other side, $B(\omega)$ is of the form

$$B(\omega) = \mathrm{e}^{\mathrm{i}\omega n_0} \sum_{n=0}^{N-1} b[n]\, \mathrm{e}^{-\mathrm{i}\omega n} = \mathrm{e}^{\mathrm{i}\omega n_0} b[N-1] \prod_{j=1}^{N-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j\right).$$

- $P_A$ can also be written as

$$\begin{aligned} P_A\left(\mathrm{e}^{-\mathrm{i}\omega}\right) &= \mathrm{e}^{-\mathrm{i}\omega(N-1)} A(\omega) = \mathrm{e}^{-\mathrm{i}\omega(N-1)} \left|B(\omega)\right|^2 \\ &= \left|b[N-1]\right|^2 \prod_{j=1}^{N-1} \overline{\beta}_j \prod_{j=1}^{N-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j\right)\left(\mathrm{e}^{-\mathrm{i}\omega} - \overline{\beta}_j^{-1}\right). \end{aligned}$$

# Nontrivial ambiguities

(Real case: BRUCK, SODIN [1979])

## Theorem

*Let $A(\omega)$ be a nonnegative trigonometric polynomial with $a[-n] = \overline{a[n]}$. Then the problem*

$$\left|B(\omega)\right|^2 = A(\omega)$$

*has at least one solution.*

*Furthermore, each solution can be written in the form*

$$B(\omega) = \mathrm{e}^{\mathrm{i}\alpha + \mathrm{i}\omega n_0} \left[ \left|a[N-1]\right| \prod_{j=1}^{N-1} \left|\beta_j\right|^{-1} \right]^{\frac{1}{2}} \prod_{j=1}^{N-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j\right).$$

*Assuming that $A(\omega) = \mathrm{e}^{\mathrm{i}\omega(N-1)} P_A(\mathrm{e}^{-\mathrm{i}\omega})$, where $P_A(z)$ is a polynomial with paired zeros $\gamma_j$ and $\overline{\gamma}_j^{-1}$, we can choose $\beta_j \in \left\{\gamma_j, \overline{\gamma}_j^{-1}\right\}$.*

# Convolution

**Definition (Convolution of signals)**

$$(x_1 * x_2)[n] := \sum_{k \in \mathbb{Z}} x_1[k]\, x_2[n-k].$$

**Theorem (BEINERT, PLONKA-HOCH [2014])**

*Let* $\left|\widehat{x}(\omega)\right|^2$ *be given. Further, let* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *be two finite signals with*

$$\mathbf{x} = \mathbf{x}_1 * \mathbf{x}_2.$$

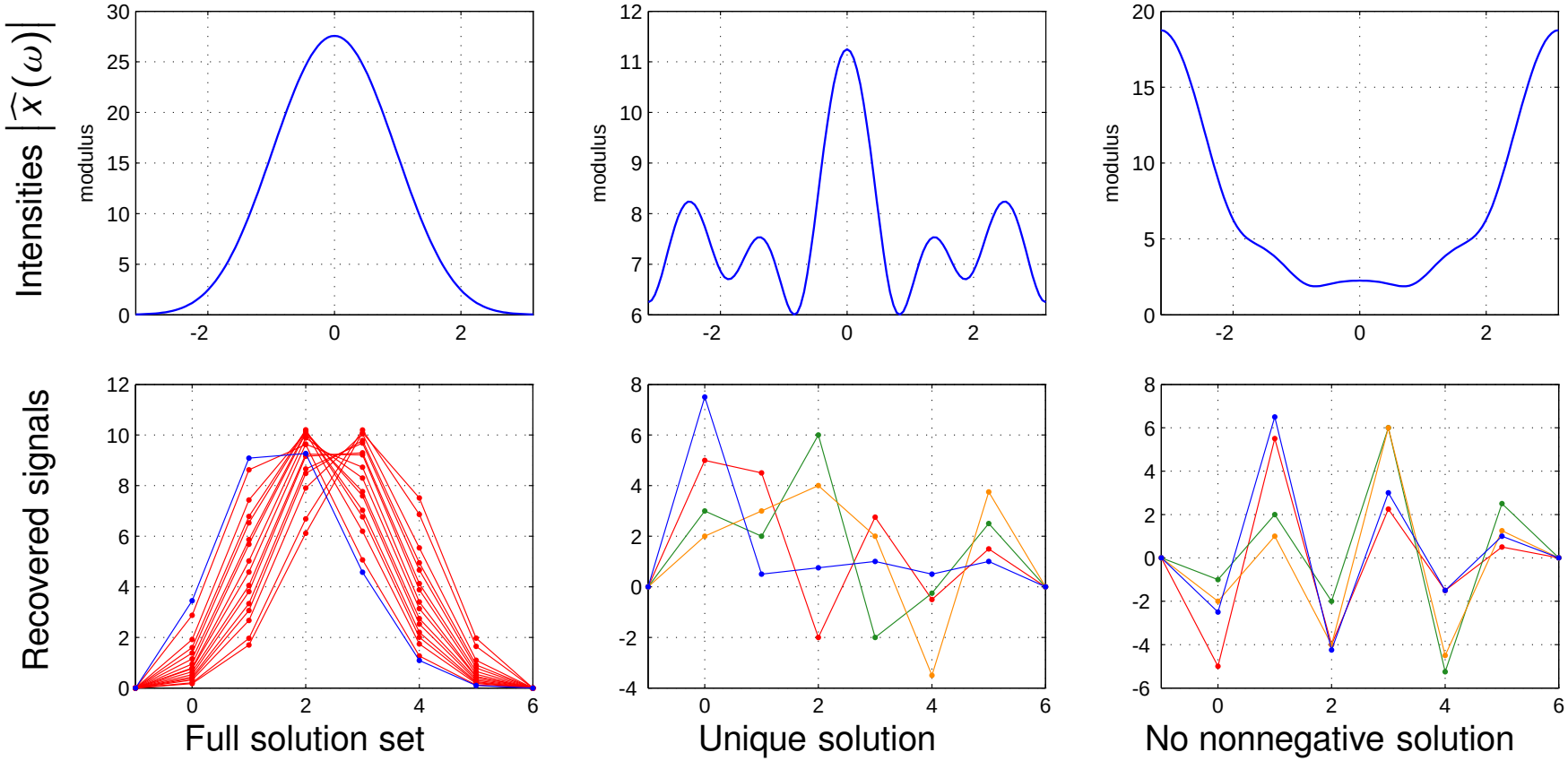
*Then*

$$\mathbf{y} := \mathrm{e}^{\mathrm{i}\alpha}\left(\overline{x_1[-\cdot]}\right) * \left(x_2[\cdot - n_0]\right)$$

*has the same* FOURIER *intensities. Moreover, all signals with the* FOURIER *intensities* $\left|\widehat{x}(\omega)\right|^2$ *have such a representation.*

# Positivity of the real signal

## Example



# Additional end points

(Real case: XU, YAN, CHANG [1987])

- Assume that the end point of the signal

$$x[N-1] = \mathrm{e}^{\mathrm{i}\alpha}\left[\left|a[N-1]\right|\prod_{j=1}^{N-1}\left|\beta_j\right|^{-1}\right]^{\frac{1}{2}}$$

  is additionally given.
- Non-uniqueness: Replacing the first $L$ zeros yields the condition

$$\prod_{j=1}^{L}\left|\beta_j\right|^2 - 1 = 0.$$

**Theorem**

*Almost every signal* $\mathbf{x}$ *can be uniquely recovered from its* FOURIER *intensities* $\left|\widehat{x}(\omega)\right|^2$ *and its end point* $x[N-1]$.

# Given moduli of the signal

(SEIFERT, STOLZ, DONATELLI, LANGEMANN, TASCHE [2006]; LANGEMANN, TASCHE [2008])

- Assume that the modulus of the signal $\left|x[n]\right|^2$ is additionally given.
- Non-uniqueness: Again we get for $x[N-1]$ the condition

$$\prod_{j=1}^{L} \left|\beta_j\right|^2 - 1 = 0.$$

**Theorem**

*Almost every signal* $\mathbf{x}$ *can be uniquely recovered from the* FOURIER *intensities* $\left|\widehat{x}(\omega)\right|^2$ *and the moduli* $\left|x[n]\right|^2$ *up to rotations.*

# Interference with a known signal

(Real Case: KIM, HAYES [1990])

- Let **h** be a known finite length signal.
- Assume that the FOURIER intensities $\left|\widehat{y}(\omega)\right|^2$ of

$$y[n] = x[n] + h[n] \qquad (n \in \mathbb{Z})$$

  are also given.
- In some cases, e.g. if **h** is a DIRAC signal or has linear phase, the phase retrieval problem is uniquely solvable.

# Interference with an unknown signal

(Real case: KIM, HAYES [1993]; Complex case: RAZ, DUDOVICH, NADLER [2013])

**Theorem**

*Let* $\mathbf{x}$ *and* $\mathbf{h}$ *be two complex signals with finite support and assume that the factorizations of their symbols*

$$\widehat{x}(\omega) = \mathrm{e}^{\mathrm{i}\omega n_1}\, x[N_1 - 1] \prod_{j=1}^{N_1-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \eta_j\right),$$

$$\widehat{h}(\omega) = \mathrm{e}^{\mathrm{i}\omega n_2}\, h[N_2 - 1] \prod_{j=1}^{N_2-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right)$$

*have no common nonzero roots. Then* $\mathbf{x}$ *and* $\mathbf{h}$ *can be uniquely recovered from* $\left|\widehat{x}(\omega)\right|^2$, $\left|\widehat{h}(\omega)\right|^2$ *and* $\left|\widehat{x}(\omega) + \widehat{h}(\omega)\right|^2$ *up to trivial ambiguities.*

# Interference with an unknown signal

**Proof.**

- Assume the problem has two solutions $x[n]$, $h[n]$ and $\widetilde{x}[n]$, $\widetilde{h}[n]$.
- Use the factorizations

$$\widehat{x}(\omega) = \mathrm{e}^{\mathrm{i}\omega n_1}\,\widehat{x_1}(\omega)\,\widehat{x_2}(\omega) \quad \text{and} \quad \widehat{\widetilde{x}}(\omega) = \mathrm{e}^{\mathrm{i}\alpha_1}\mathrm{e}^{\mathrm{i}\omega k_1}\,\widehat{x_1}(\omega)\,\overline{\widehat{x_2}(\omega)},$$

$$\widehat{h}(\omega) = \mathrm{e}^{\mathrm{i}\omega n_2}\,\widehat{h_1}(\omega)\,\widehat{h_2}(\omega) \quad \text{and} \quad \widehat{\widetilde{h}}(\omega) = \mathrm{e}^{\mathrm{i}\alpha_2}\mathrm{e}^{\mathrm{i}\omega k_2}\,\widehat{h_1}(\omega)\,\overline{\widehat{h_2}(\omega)}.$$

- Consider the identity

$$\left|\widehat{x}(\omega) + \widehat{h}(\omega)\right|^2 = \left|\widehat{\widetilde{x}}(\omega) + \widehat{\widetilde{h}}(\omega)\right|^2.$$

□

# Summary/Outlook

- Characterization of the ambiguities in the one-dimensional phase retrieval problem.
- Quality of preconditions and additional data.

- Phase retrieval in higher dimension.
- Ambiguities in the FRESNEL regime.

Thank you for your attention.

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