Ambiguities in one-dimensional phase retrieval of structured functions

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Abstract: Phase retrieval means that we wish to recover a complex-valued signal (discrete or continuous) from the magnitudes of its Fourier transform. Here we restrict ourselves to the recovery of structured functions, e.g. linear spline functions with equidistant knots. First, a complete characterization of the occurring ambiguities is presented. Moreover, we investigate additionally given moduli of the signal itself regarding their ability to reduce the set of ambiguities.

1. Introduction

The phase retrieval problem appears in a wide range of applications in physics, e.g. in crystallography, electron microscopy, astronomy, and optics. Here we wish to recover an unknown structured function f of the form

$$f(t) := \sum_{n \in \mathbb{Z}} c[n]\, \phi(t - n) \tag{1.1}$$

from its Fourier intensities $|\widehat{f}|$ given by

$$\left|\widehat{f}(\omega)\right| = \left| \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt \right|, \tag{1.2}$$

where ϕ is an a priori given generator function in $C^0 \cap L^1$ not equal to zero, and where $(c[n])_{n \in \mathbb{Z}}$ is the complex-valued sequence with finite support. If we choose the centred linear B-spline $\phi(t) := (1 - |t|)\, \chi_{[-1,1]}(t)$ as generator function, then the unknown function f is a linear spline function. This specific phase retrieval problem was introduced in [LT08, SSD$^+$06]. Obviously, the considered phase retrieval problem is not uniquely solvable. For example, we can construct further solutions simply by rotating (multiplying a unimodular constant), shifting, or reflecting and conjugating a given solution. Moreover, besides these trivial ambiguities, there exist further non-trivial ambiguities.

2. Phase retrieval of discrete-time signals

In order to characterize all ambiguities – trivial and non-trivial – of the phase retrieval problem to recover a structured function, we can use our findings in [BP15], where we consider the problem to reconstruct an unknown complex-valued signal $x := (x[n])_{n \in \mathbb{Z}}$ with finite support from its discrete-time Fourier intensities $|\widehat{x}|$ given by $|\widehat{x}(\omega)| = |\sum_{n \in \mathbb{Z}} x[n] \, \mathrm{e}^{-\mathrm{i}\omega n}|$. Denoting the support length of the signal x by N, we introduce the autocorrelation signal and the corresponding autocorrelation polynomial defined by

$$a[n] := \sum_{k \in \mathbb{Z}} x[k] \, \overline{x[k+n]} \qquad \text{and} \qquad P(z) := z^{N-1} \sum_{n=-N+1}^{N-1} a[n] \, z^n \qquad (2.3)$$

respectively, where $(a[n])_{n \in \mathbb{Z}}$ is known from $|\widehat{x}(\omega)|^2 = \sum_{n \in \mathbb{Z}} a[n] \, \mathrm{e}^{-\mathrm{i}\omega n}$. Now, we can characterize all ambiguities in the following manner, cf. [BP15, Theorem 2.4].

Theorem 2.1. *Let x be a discrete-time signal with finite support length N. Then the Fourier transform of each discrete-time signal y satisfying $|\widehat{y}| = |\widehat{x}|$ can be written in the form*

$$\widehat{y}(\omega) = \mathrm{e}^{\mathrm{i}(\alpha + \omega n_0)} \sqrt{|a[N-1]| \prod_{j=1}^{N-1} |\beta_j|^{-1}} \cdot \prod_{j=1}^{N-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j \right)$$

where $\alpha \in [-\pi, \pi)$, $n_0 \in \mathbb{Z}$, and β_j is chosen from the zero pairs $(\gamma_j, \overline{\gamma}_j^{-1})$ of the autocorrelation polynomial P.

There have been many different approaches, such as additional interference measurements [KH90, KH93], to enforce uniqueness of the solution or to reduce the set of non-trivial solutions of the one-dimensional phase retrieval problem for real-valued discrete-time signals. Using Theorem 2.1, we can now examine the quality of these approaches also for the recovery of complex-valued discrete-time signals, see [BP15]. In some applications, besides the moduli of the Fourier transform, the moduli of the unknown complex-valued signal itself are known. With these additional constraints, the set of non-trivial ambiguities is almost always reduced to one unique solution up to trivial ambiguities caused by rotations, see [BP15, Section 3.4]. More precisely, we have the following result.

Theorem 2.2. *Almost every signal x can be uniquely recovered from its Fourier intensities $|\widehat{x}|$ and the moduli $|x|$ of the signal itself up to rotations.*

3. Phase retrieval of structured functions

Now, we want to apply the result in Theorem 2.1 on discrete-time signals in order to characterize all ambiguities in the recovery of a structured function f of the form (1.1) from its Fourier intensities.

> **Corollary 3.1.** *Let f be a structured function* (1.1) *with finite support length N of the corresponding coefficients sequence* $(c[n])_{n \in \mathbb{Z}}$. *Then the Fourier transform of each structured function g satisfying* $|\widehat{g}| = |\widehat{f}|$ *can be written in the form*
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> $$\widehat{g}(\omega) = e^{i(\alpha + \omega n_0)} \, \widehat{\phi}(\omega) \sqrt{|a[N-1]| \prod_{j=1}^{N-1} |\beta_j|^{-1}} \cdot \prod_{j=1}^{N-1} \left(e^{-i\omega} - \beta_j \right)$$
>
> *where $\alpha \in [-\pi, \pi)$, $n_0 \in \mathbb{Z}$, and β_j is chosen from the zero pairs $(\gamma_j, \overline{\gamma}_j^{-1})$ of the autocorrelation polynomial P in* (2.3) *with the autocorrelation sequence* $(a[n])_{n \in \mathbb{Z}}$ *determined by* $\sum_{n \in \mathbb{Z}} a[n] \, e^{-i\omega n} = |\widehat{f}(\omega)/\widehat{\phi}(\omega)|^2$ *whenever $\widehat{\phi}(\omega) \neq 0$.*

Proof. Let g defined by $g(t) := \sum_{n \in \mathbb{Z}} b[n] \, \phi(t - n)$ be a structured function with the same Fourier intensities as f. Hence, the squared Fourier intensities of the structured function f and g have to fulfil the equation

$$\left| \widehat{f}(\omega) \right|^2 = \left| \widehat{\phi}(\omega) \right|^2 \left| \sum_{n \in \mathbb{Z}} c[n] \, e^{-i\omega n} \right|^2 = \left| \widehat{\phi}(\omega) \right|^2 \left| \sum_{n \in \mathbb{Z}} b[n] \, e^{-i\omega n} \right|^2 = |\widehat{g}(\omega)|^2 \,.$$

Since the squared discrete-time Fourier intensities of the coefficients sequences $c := (c[n])_{n \in \mathbb{Z}}$ and $b := (b[n])_{n \in \mathbb{Z}}$ are non-negative trigonometric polynomials (the sequences have only a finite support), and since $\widehat{\phi}$ is continuous and not zero everywhere, the squared Fourier intensities of the signals c and b can be completely determined by the intensities $|\widehat{f}|$ or $|\widehat{g}|$.

Therefore, the phase retrieval problem for the structured function f can be reduced to the phase retrieval problem to recover the coefficients sequence $(c[n])_{n \in \mathbb{Z}}$ from the corresponding discrete-time Fourier intensities. Thus, the occurring ambiguities in the recovery of the coefficients sequence with finite support are characterized by Theorem 2.1.
□

Finally, we are interested in reducing the ambiguities of the phase retrieval problem for structured functions. Because of the close relation to the phase retrieval problem for discrete-time signals, the ideas in [BP15] to employ additional interference measurements for ambiguity reduction can be simply transferred. For Theorem 2.2, however, we need further appropriate side conditions. More precisely, we assume that the generator function ϕ is a Lagrange function, i.e. $\phi(n) = \delta(n)$ for all $n \in \mathbb{Z}$, cf. [LT09].

Corollary 3.2. *Let ϕ be a Lagrange function. Then almost every structured function f can be uniquely recovered from its Fourier intensities $|\widehat{f}|$ and the moduli $|f(n)|$ for $n \in \mathbb{Z}$ up to rotations.*

Proof. With the identity $|f(n)| = |c[n]|$, the considered problem can be reduced to the recovery of the coefficients sequence $c = (c[n])_{n \in \mathbb{Z}}$ from the Fourier intensities $|\widehat{c}|$ and the moduli $|c|$. Now, the application of Theorem 2.2 yields the assumption. $\square$

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